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OPTIMAL ASSET ALLOCATION BASED ON UTILITY MAXIMIZATION IN THE PRESENCE OF MARKET FRICTIONS

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Abstract

We develop a model of optimal asset allocation based on a utility framework. This applies to a more general context than the classical mean-variance paradigm since it can also account for the presence of constraints in the portfolio composition. Using this approach, we study the distribution of a measure of wealth compensative variation, we propose a benchmark and portfolio efficiency test and a procedure to estimate the implicit risk aversion parameter of a power utility function. Our empirical analysis makes use of the S&P 500 and industry portfolios time series to show that although the market index cannot be considered an efficient investment in the mean-variance metric, the wealth loss associated with such an investment is statistically different from zero but rather small (lower than 0.5%). The wealth loss is at its minimum for a representative agent with a constant risk aversion index not higher than 5. Furthermore we show that, for reasonable levels of risk aversion, the use of an equally weighted portfolio is surprisingly consistent with an expected utility maximizing behavior.

JEL classification codes: C15, D14, G11

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1. Introduction

The efficiency of an investment is usually assessed by means of a standard mean-variance approach. In the simplest case of no restrictions on portfolio shares, such a framework implies that the performance of any investment is measured in terms of its Sharpe ratio, i.e., the expected return over the standard deviation of its excess returns. Using such a measure, several statistical tests have been developed to establish the efficiency of an investment; among others, the tests proposed by Jobson and Korkie (1982), Gibbons et al. (1989), and Gourieroux and Jouneau (1999) are noteworthy.

The use of the Sharpe ratio is relatively simple and rather intuitive but lacks some important features. The most important being that, by acting this way, it is not possible to take account of market imperfection when building the optimal portfolio weights. The widespread use of Sharpe ratios depends on the well-known fact that their upper limit is reached by any portfolio in the mean-variance efficient frontier built as a combination of the market portfolio and the risk free asset. Such a frontier is derived disregarding market imperfections, but in their presence it would take a different shape.

In particular, two kinds of constraints are relevant: transaction costs and inequality constraints. Transaction costs are costs incurred when buying or selling assets. These include brokers' commissions and spreads, i.e., the difference between the price paid for an asset and the price it can be sold. Transaction costs may be negligible in the case of financial assets, but several authors (among others Grossman and Laroque, 1990, Flavin, 2002, and Pelizzon and Weber, 2003) point out how they are instead relevant for real assets such as housing[1]. Following Gourieroux and Jouneau (1999) we know that, when equality constraints on some portfolio weights are taken into account, it is however possible to translate the original plane in another mean-variance frontier, conditional on the constrained assets.

Another important market imperfection is represented by inequality constraints. In actual stock markets, for instance, short sales are not prohibited, but discouraged by the fact that the proceeds are not normally available to be invested elsewhere; this is enough to eliminate a private investor with just mildly negative beliefs (Figlewski, 1981). On the contrary, mutual fund constraints are widespread and may be seen as one component of the set of monitoring mechanisms that reduce the costs arising from frictions in the principal-agent relation (Almazan et al., 2004). Considering these constraints, we would be faced with a different frontier of feasible portfolios, of unknown shape, whose relationship with the Sharpe ratio is not clear. With only short-sale restrictions in particular, there may be switching points along the mean-variance frontier corresponding to changes in the set of assets held. Each switching

[1] to such an extent that real assets become illiquid in the presence of transaction costs. In other words, their investment is kept as fixed in the short run, and an optimizing investor chooses the composition of her financial portfolio conditional on the stock held in real assets.

point corresponds to a kink (Dybvig, 1984), and the mean-variance frontier consists then of parts of the unrestricted mean-variance frontiers computed on subsets of the primitive assets.

Notwithstanding this evidence, empirical works often come out with optimal portfolio weights in a standard mean-variance framework that take extreme values (both negative and positive) in some assets. Green and Hollifield (1992) state that: *«[…] The extreme weights in efficient portfolios are due to the dominance of a single factor in the covariance structure of returns, and the consequent high correlation between naively diversified portfolios. With small amounts of cross-sectional diversity in asset betas, well-diversified portfolios can be constructed on subsets of the assets with very little residual risk and different betas. A portfolio of these diversified portfolios can then be constructed that has zero beta, thus eliminating the factor risk as well as the residual risk»*. This portfolio is unfeasible in practice and, unjustifiably, gets compared with observed investments in terms of Sharpe ratios[2]. This way, we relate actual investments with unrealistic ones, which ensure an even better performance than the optimal feasible portfolios. Hence, the comparison is erroneous since it tends to overestimate the inefficiency of any observed investment.

The problem is dealt with in Basak et al. (2002) and Bucciol (2003); following a mean-variance approach, these authors develop an efficiency test in which the discriminating measure is no longer based on a Sharpe ratio comparison, but on a variance comparison instead, for a given expected return. Such a technique, nevertheless, circumvents the above mentioned problem at the cost of neglecting some information: it just fixes the value of the expected return, and does not take into account how it could affect the importance of deviations in risk.

In this paper we try, instead, to cope with inequality constraints in a model that pays attention to expected returns as well as variance of investment returns. In lieu of working with efficient frontiers, we concentrate on the expected utility paradigm. Quoting Gourieroux and Monfort (2005), *«the main arguments for adopting the mean-variance approach and the normality assumption for portfolio management and statistical inference are weak and mainly based on their simplicity of implementation»*. It is well known (Campbell and Viceira, 2002), however, that the two procedures provide the same results, under several assumptions. Already Brennan and Torous (1999), Das and Uppal (2004) and Gourieroux and Monfort (2005) consider an agent who maximizes her expected utility in order to get an optimal portfolio. Brennan and Torous (1999), in particular, define a performance measure, based on the concept of compensative variation, which compares the utility from an optimal investment with that resulting from a given investment. Drawing inspiration from this strand

[2] Any portfolio is indeed proportional to the zero-beta portfolio since the two fund separation theorem holds.

of literature we will subsequently show that, using a specific utility function, this procedure boils down to maximizing a function of mean and variance of a portfolio, for a given risk aversion; furthermore, the measure of compensative variation has the intuitive economic interpretation of the amount of wealth wasted or generated by the investment, relative to the optimal portfolio. The main contribution of this paper is to characterize the asymptotic probability distribution and confidence intervals of this measure of compensative variation; this will permit us to conduct statistically valid inference, and therefore to test for portfolio or benchmark efficiency. This task is made difficult, nevertheless, by the presence of inequality constraints.

The paper is organized as follows: section 2 compares the standard mean-variance approach with our approach based on expected utility maximization. It shows the underlying algebra of the agent's problem, and introduces a measure of wealth compensative variation. Section 3 specifies the efficiency test, by means of a weak version of the central limit theorem and the delta method. This procedure does not permit to run the test for extreme null hypotheses (e.g., all the wealth is wasted), but is enough to construct confidence intervals. Section 4 describes the statistic in a closed-form expression when there are no inequality constraints, and examines analogies with optimal portfolios derived in a mean-variance framework. Section 5 presents a fruitful way to estimate the relative risk aversion parameter using the data. In the absence of constraints, the expression can be derived in a clear closed-form expression; otherwise it can be obtained numerically. In section 6 we describe the data used in the empirical exercise, the S&P 500 index and 10 industry portfolios for the U.S. market. We further run some tests to assess the efficiency of the S&P index, the unconstrained optimal portfolio or a naïve portfolio; we also compute the optimal risk aversion parameters. In section 7 we study the empirical distribution of our test, running several Monte Carlo simulations. Lastly, section 8 summarizes the results and concludes.

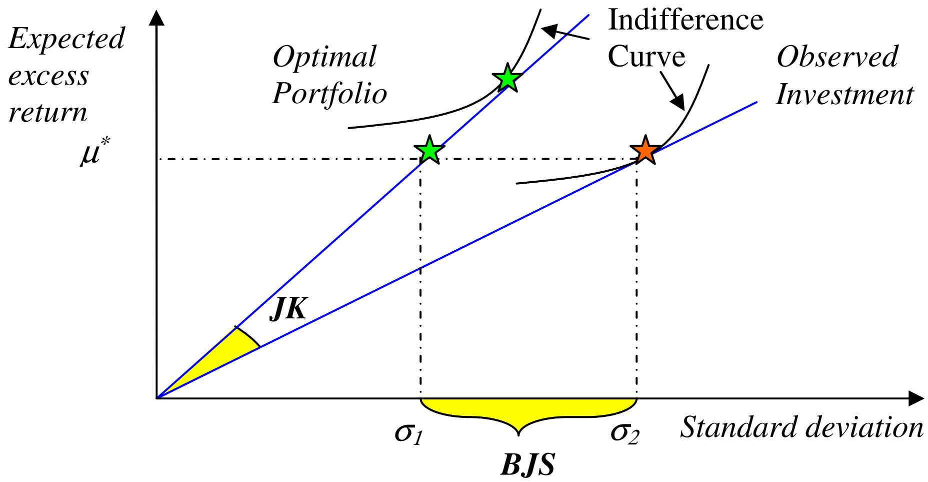
2. Agent's behavior

Disregarding constraints, we may assess the efficiency of an investment by comparing its Sharpe ratio with the optimal, as shown in figure 1[3]. It is the case, for instance, of the test proposed by Jobson and Korkie (henceforth JK, 1982) in a portfolio setting. The optimal Sharpe ratio depicts the slope of the efficient frontier which includes a risk free asset within the endowment. The greater the difference between the two ratios, the greater the inefficiency of the observed investment (figure 1).

[3] Although in the figure we draw an optimal portfolio with the same expected excess return as the observed investment, there are infinite optimal portfolios with the same Sharpe ratio; they differ only in the share invested in the risk free asset.

Figure 1.

Measures of efficiency – mean-variance framework



Some other tests, such as the one in Basak, Jagannathan and Sun (BJS, 2002), fix the level of expected return μ^* and consider the difference between the two variances, σ_1^2 and σ_2^2, namely the lowest achievable variance minus the observed variance. The smaller this difference (negative by construction), the higher the inefficiency of the observed investment. A caveat of this approach is that one dimension of the problem, the expected excess return, is kept fixed and therefore completely neglected by the efficiency analysis. It is however difficult to think of different ways to face this problem, since the shape of the efficient frontier does not admit a closed-form representation in the presence of inequality constraints.

A reasonable alternative is to consider an expected utility framework instead of a mean-variance approach. It is well known that the two methods are equivalent under several assumptions; Campbell and Viceira (2002), for instance, argue that a power (or CRRA) utility function and log-normally distributed asset returns produce results that are consistent with those of a standard mean-variance analysis. The property of constant relative risk aversion, moreover, is attractive and helps explain the stability of financial variables over time.

We then draw inspiration from Gourieroux and Monfort (2005) and study the economic behavior of a rational agent who maximizes her expected utility of future wealth. The authors explain that such an approach is appropriate even when return distributions do not seem normal; in our context, this framework also takes account of constraints in portfolio composition.

In figure 1 the indifference curves for observed and optimal portfolios is drawn. The optimal portfolio does not need to be the same as the one in the mean-variance framework; we know (see §4) that, in the absence of constraints, it differs only in how much is invested in the risk free component. Our test, then, accounts for the distance between the two indifference curves; the greater the distance, the greater the inefficiency. The reason why we base our work on this measure is that, in the presence of market frictions, it is no longer true that the Sharpe ratio is an adequate quantity to assess the efficiency and, at the same time, the simple difference between variances considers just part of the available information.

Brennan and Torous (1999) analyze the same problem in a portfolio choice framework with a power utility function and come up with a measure of compensative variation which calculates the amount of wealth wasted when adopting a suboptimal portfolio allocation strategy; the same concept is used in Das and Uppal (2004) when assessing the relevance of systemic risk in portfolio choice.

In the following sections we show how this measure of compensative variation can be used to develop an efficiency test whose validity is not affected by the presence of equality and/or inequality constraints on the portfolio asset shares.

2.1. An approach based on utility comparison

According to Brennan and Torous (1999), an investor is concerned with maximizing the expected value of a power utility function defined over her wealth at the end of the next period:

$$U\left(W_{t+dt}\right)=\frac{W_{t+dt}^{1-\gamma}-1}{1-\gamma}$$

where $\gamma>0$ is the relative risk aversion (RRA) coefficient and W_{t+dt} the wealth at time $t+dt$.

Our investor holds a benchmark b [4]. We assume that the price P_t^b at time t of the benchmark follows the stochastic differential equation

$$\frac{dP_t^b}{P_t^b}=\mu_b dt+\sigma_b d\beta_t^b=\left(\eta_b+r_0\right)dt+\sigma_b d\beta_t^b \tag{1}$$

where μ_b (expected return) and σ_b (standard deviation) are constants, and $d\beta_t^b$ is the increment to a univariate Wiener process. In this framework, the overall wealth W_t evolves with P_t^b:

$$\frac{dW_t}{W_t}=\frac{dP_t^b}{P_t^b}$$

[4] A standard against which the performance of a security, index or investor can be measured. We use this term according to Basak et al. (2002), but we could instead consider a mutual fund, a pension fund etc.

Using a property of the geometric Brownian motion, equation (1) implies that, over any finite interval of time $\left[t, t+dt\right]$

$$W_{t+dt} = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)(t+dt-t)+\sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)} = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt+\sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)}$$

with $\beta_t^b \sim N(0,t)$. In turn this implies that W_{t+dt} is conditionally log-normally distributed:

$$W_{t+dt} \mid W_t \sim LN\left(\log\left(W_t\right) + \left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt, \sigma_b^2 dt\right)$$

with expectation

$$E\left[W_{t+dt} \mid W_t\right] = W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt} E\left[e^{\sigma_b\left(\beta_{t+dt}^b - \beta_t^b\right)}\right] =$$

$$= W_t e^{\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt} e^{\frac{1}{2}\sigma_b^2(t+dt-t)} = W_t e^{\mu_b dt}$$

Therefore, the expected utility associated with the benchmark is given by

$$E\left[U\left(W_{t+dt}\right) \mid \mu_b, \sigma_b, \gamma, W_t\right] = \frac{1}{1-\gamma}\left(E\left[W_{t+dt}^{1-\gamma} \mid W_t\right] - 1\right) = \frac{1}{1-\gamma}\left(E\left[\left(e^{\log W_{t+dt}}\right)^{1-\gamma} \mid W_t\right] - 1\right) =$$

$$= \frac{1}{1-\gamma}\left(E\left[e^{(1-\gamma)\log W_{t+dt}} \mid W_t\right] - 1\right) = \frac{1}{1-\gamma}\left(e^{(1-\gamma)\log W_t + (1-\gamma)\left(\mu_b - \frac{1}{2}\sigma_b^2\right)dt + \frac{1}{2}\sigma_b^2(1-\gamma)^2 dt} - 1\right) =$$

$$= \frac{1}{1-\gamma}\left(W_t^{1-\gamma} e^{(1-\gamma)\mu_b dt - \frac{1}{2}\sigma_b^2\gamma(1-\gamma)dt} - 1\right) = \frac{1}{1-\gamma}\left(W_t^{1-\gamma} e^{(1-\gamma)\left(\mu_b - \frac{1}{2}\gamma\sigma_b^2\right)dt} - 1\right)$$

In order to study the efficiency of such an investment, an investor compares its performance with that of the best alternative: a portfolio of primitive assets. The endowment is given by one risk free asset (with return r_0) and a set of n risky assets (with return $r_i, i = 1, ..., n$).

Calling w_i the fraction of wealth allocated to the *i*-eth risky asset, w the vector of w_i's and $(1 - w'\iota)$ the residual fraction invested in the risk free asset, the overall wealth evolves as

$$\frac{dW_t}{W_t} = \mu_p dt + \sigma_p d\beta_t$$

where $d\beta_t$ is the increment to an univariate Wiener process, and $\left(\mu_p, \sigma_p^2\right)$ are the first two moments of the portfolio:

$$\mu_p = w'\left(\mu - r_0\iota\right) + r_0 = w'\eta + r_0 = \eta_p + r_0$$

$$\sigma_p^2 = w'\Sigma w$$

and μ and Σ are the vector of the expected returns and the covariance matrix, respectively.
Following the computation already made for the benchmark case, the expected utility is

$$E\left[U\left(W_{t+1}\right)\mid \mu_p,\sigma_p,\gamma,W_t\right]=\frac{1}{1-\gamma}\left(W_t^{1-\gamma}e^{(1-\gamma)\left(\mu_p-\frac{1}{2}\gamma\sigma_p^2\right)dt}-1\right)$$

We consider a "buy & hold" strategy in which the investor observes the asset returns at time t and makes her choice once and forever; it is intended to represent the type of inefficiency in portfolio allocations induced by the *status quo* bias described in Samuelson and Zeckhauser (1988).
The optimal portfolio w^* is defined as

(2) $$w^*=\arg\max_w E\left[U\left(W_{t+1}\right)\mid \mu_p,\sigma_p,\gamma,W_t\right]$$

subject to several constraints (equality, inequality, sum to one etc.) on its composition:

$$Aw=a$$

$$lb\le w\le ub$$

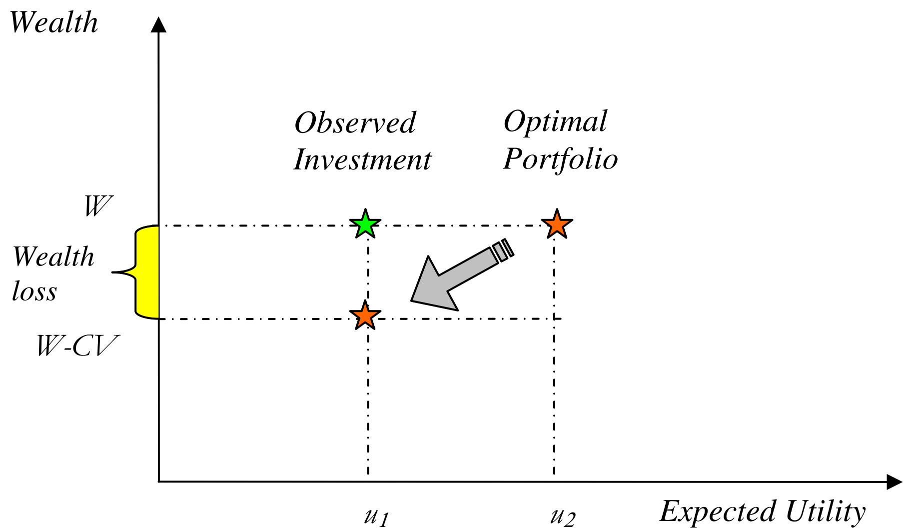
A natural way to assess the performance of the benchmark, then, is to compare its expected utility with that resulting from the optimal portfolio.
In accordance with Brennan and Torous (1999) and Das and Uppal (2004), we establish this comparison by means of a compensative variation metric. In other words, we pose the question of what level of initial wealth W_t^* is needed to obtain with the optimal portfolio the same expected utility as with the benchmark and initial wealth W_t. This technique is graphically described in figure 2 and, in formulae, in the equation

$$E\left[U\left(W_{t+1}\right)\mid \mu_p,\sigma_p,\gamma,W_t^*\right]=E\left[U\left(W_{t+1}\right)\mid \mu_b,\sigma_b,\gamma,W_t\right]$$

where we want to derive $W_t^*=W_t-CV$, with CV amount of wealth wasted (if positive) or generated (if negative) by the benchmark instead of using the best alternative.

Figure 2.

Measures of efficiency – expected utility framework



Note: the figure shows the case $CV > 0$ only.

Therefore,

$$\frac{\left(W_t - CV\right)^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\left(\mu_p - \frac{1}{2}\gamma\sigma_p^2\right)dt} = \frac{W_t^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\left(\mu_b - \frac{1}{2}\gamma\sigma_b^2\right)dt}$$

so that

$$CV = W_t \left[1 - \exp\left\{ \left(\mu_b - \frac{1}{2}\gamma\sigma_b^2 \right) dt - \left(\mu_p - \frac{1}{2}\gamma\sigma_p^2 \right) dt \right\} \right]$$

or, in relative terms,

$$\begin{aligned} cv_b = cv\left(\eta_b, \sigma_b^2, \eta, \Sigma, \gamma\right) = \frac{CV}{W_t} &= \left[1 - \exp\left\{ \left(\mu_b - \frac{1}{2}\gamma\sigma_b^2 \right) dt - \left(\mu_p - \frac{1}{2}\gamma\sigma_p^2 \right) dt \right\} \right] \\ &= \left[1 - \exp\left\{ \left(\eta_b - \frac{1}{2}\gamma\sigma_b^2 \right) dt - \left(\eta_p - \frac{1}{2}\gamma\sigma_p^2 \right) dt \right\} \right] = \\ &= \left[1 - \exp\left\{ \left(\eta_b - \frac{1}{2}\gamma\sigma_b^2 \right) dt - \left(w^{*\prime}\eta - \frac{1}{2}\gamma w^{*\prime}\Sigma w^{*} \right) dt \right\} \right] \end{aligned}$$

with $cv_b \in (-\infty, 1]$. This function has a clear economic interpretation: it measures the amount of wealth that the agent wastes (if positive) or generates (if negative) with respect to the initial level of wealth, when using the benchmark instead of the best alternative. $cv_b = 1$ means that the benchmark is completely inefficient (the agent is wasting 100 percent of her wealth); $cv_b \to -\infty$, instead, means that the benchmark is totally efficient (the agent is generating infinite new wealth).

In case we want to assess the efficiency of a portfolio ω, instead of a benchmark, against the optimal portfolio w, it is easily shown that the relative wealth loss is

$$cv_p = cv(\eta, \Sigma, \gamma) = \frac{CV}{W_t} = \left[1 - \exp\left\{\left(\omega'\eta - \frac{1}{2}\gamma\omega'\Sigma\omega\right)dt - \left(w^{*\prime}\eta - \frac{1}{2}\gamma w^{*\prime}\Sigma w^{*}\right)dt\right\}\right]$$

with $cv_p \in [0,1]$ since the observed portfolio ω comes from the *same* space of primitive assets as the optimal portfolio w^*. When $cv_p = 0$ the agent is investing in a portfolio that does not waste any wealth; it is, in other words, efficient.

We are able to associate to cv_b and cv_p a standard error, a confidence interval and an efficiency test. This will be shown in the next section, referring primarily to the benchmark case. Before proceeding with the algebra, it will nevertheless turn useful to define a simpler expression:

$$\begin{aligned} \lambda_b = \lambda(\eta_b, \sigma_b^2, \eta, \Sigma, \gamma) &= -\frac{1}{dt}\log(1 - cv_b) = \\ &= \left(w^{*\prime}\eta - \frac{1}{2}\gamma w^{*\prime}\Sigma w^{*}\right) - \left(\eta_b - \frac{1}{2}\gamma\sigma_b^2\right) = \\ &= \max_w \left\{\left(w'\eta - \frac{1}{2}\gamma w'\Sigma w\right) - \left(\eta_b - \frac{1}{2}\gamma\sigma_b^2\right)\right\} \end{aligned} \tag{3}$$

in the case of a benchmark, and likewise λ_p for the portfolio.

It is worth pointing out that the optimal weights w^* in the agent's problem (2) are the same as we would get by maximizing λ_b or λ_p in (3) subject to the same constraints. From the investor's point of view, therefore, maximizing the expected utility or its transformation is equivalent.

Below we ignore the constant term that involves dt [5], for the sake of simplicity and since it disappears when computing the test statistic.

3. Development of an efficiency test

The function $\lambda(\eta_b, \sigma_b^2, \eta, \Sigma, \gamma)$ depends on unknown moments[6] and has to be replaced by a consistent sample estimate, defined as

[5] The reader can assume that $dt = 1$.

[6] Let us assume for now to know the relative risk aversion coefficient γ.

$$\ell_b = \ell\left(e_b, s_b^2, e, S, \gamma\right) = \max_w \left\{ \left(w'e - \frac{1}{2}\gamma w'Sw \right) - \left(e_b - \frac{1}{2}\gamma s_b^2 \right) \right\} \tag{4}$$

subject to the constraints

$$Aw = a$$

$$lb \le w \le ub$$

We solve, therefore, the maximization problem using a function of *sample* moments instead of *true* moments. As a consequence, we need to take account of sampling errors and derive a statistical distribution for the ℓ_b function. Yet establishing its exact distribution is both cumbersome and useless. It is cumbersome because the presence of inequality constraints hinders the recourse to standard statistical procedures; it is useless as, even if we knew the exact distribution, it would in the end be a mixture of different distributions. De Roon et al. (2001), dealing with inequality constraints, conclude that their statistic is asymptotically distributed as a mixture of χ^2 distributions. Therefore, even if we computed the exact distribution, this could be used only through numerical simulation. It would in fact be exactly the same procedure we should follow in the case of not knowing the exact distribution of ℓ_b.

Another possibility is to approximate the exact distribution by means of the delta method. Following Basak et al. (2002), we can use a weak central limit theorem to establish that the first and second moments of returns are asymptotically normally distributed; we can then calculate the derivative of ℓ_b relative to $\left(e_b, s_b^2, e, S\right)$, obtaining a first-order approximation of the exact distribution of $\ell\left(e_b, s_b^2, e, S, \gamma\right)$. The procedure is described in detail below.

First of all, we recognize that the only source of randomness in $\ell\left(e_b, s_b^2, e, S, \gamma\right)$ is given by the non-central first and second moments of the primitive assets and the benchmark. Since working with vectors is more convenient than with matrices, once we define

$$e_b = \frac{1}{T}\sum_{t=1}^{T} e_{tb}; \qquad e = \frac{1}{T}\sum_{t=1}^{T} e_t = \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} e_{t1} \\ \vdots \\ e_{tn} \end{bmatrix}$$

$$M_b = \frac{1}{T}\sum_{t=1}^{T} e_{tb}^2 = s_b^2 + e_b^2; \qquad M = \frac{1}{T}\sum_{t=1}^{T} M_t = \frac{1}{T}\sum_{t=1}^{T} e_t e_t' = S + ee'$$

we consider the vector $\overline{X}_T$ as

$$\overline{X}_T = \begin{bmatrix} e \\ e_b \\ vech(M) \\ M_b \end{bmatrix} = \frac{1}{T}\sum_{t=1}^{T} X_t = \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} e_t \\ e_{tb} \\ vech(M_t) \\ e_{tb}^2 \end{bmatrix}$$

where the operator $vech$ takes all the distinct elements in a symmetric matrix:

$$vech\left(e_t e_t'\right) = \begin{bmatrix} e_{t1}^2 & e_{t2}e_{t1} & \cdots & e_{tn}e_{t1} & e_{t2}^2 & \cdots & e_{tn}e_{t2} & \cdots & e_{tn}^2 \end{bmatrix}'$$

It is worth stressing one more time that the benchmark returns come from a different, although possibly correlated, parametric space than those for the primitive assets. As a consequence the benchmark could be either more or less efficient than the portfolio.

We require (i) $\{X_t, t \geq 1\}$ to be a sequence of stationary and ergodic random vectors with mean $E[X_t] = X$ and covariance matrix $\operatorname{cov}(X_t) = \Lambda$ with Λ non-singular; this is commonly assumed in the financial economics literature.

The expected value on $\overline{X}_T$ is, therefore, $E\left[\overline{X}_T\right] = X$ and its variance is

$$\begin{aligned}
Var\left(\overline{X}_T\right) &= Var\left(\frac{1}{T}\sum_{t=1}^{T} X_t\right) = E\left[\left(\frac{1}{T}\sum_{t=1}^{T} X_t - X\right)\left(\frac{1}{T}\sum_{t=1}^{T} X_t - X\right)'\right] = \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} E\left[(X_t - X)(X_s - X)'\right] = \\
&= \frac{1}{T^2}\left(\sum_{t=1}^{T} E\left[(X_t - X)(X_t - X)'\right] + \sum_{t=1}^{T}\sum_{s \neq t} E\left[(X_t - X)(X_s - X)'\right]\right) = \\
&= \frac{1}{T^2}\left(T\Lambda + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(E\left[(X_t - X)(X_s - X)'\right] + E\left[(X_s - X)(X_t - X)'\right]\right)\right) = \\
&= \frac{1}{T^2}\left(T\Lambda + \sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\left(Cov(X_t, X_s) + Cov(X_s, X_t)\right)\right) = \\
&= \frac{1}{T}\left(\Lambda + \sum_{t=1}^{T-1}\left(1 - \frac{t}{T}\right)\left(Cov(X_t, X_0) + Cov(X_0, X_t)\right)\right)
\end{aligned}$$

from which the long-run covariance matrix Λ_0 is

$$\Lambda_0 = \lim_{T \to \infty} T \; Var\left(\overline{X}_T\right) = \Lambda + 2\sum_{t=1}^{\infty} Cov(X_t, X_0).$$

Note, in particular, that we do not exclude *a priori* the possibility of a correlation between the benchmark and the primitive asset returns. Since the benchmark comes from a different parametric space than the primitive assets we do, however, exclude *a priori* a perfect correlation (± 1) between the benchmark and the portfolio. The benchmark, in other words, can only be *partially* tracked by a portfolio.

We require, furthermore, that (ii) $\lim_{T\to\infty} E\left[\left|X_t\right|^{2+\delta}\right] < \infty$ and $\lim_{T\to\infty} Var\left(I'S_T\right) = \infty \quad \forall t \geq 1, \ \forall I \in \Re^n$ and $\forall \delta \in (0,1)$, where $S_T = \sum_{t=1}^{T} X_t$; (iii) $\rho_I(t) = \lim_{t\to\infty} \max\left\{Corr\left(I'Y, I'Z\right)\right\} = 0 \quad \forall Y \in \sigma\left\{X_k : k \leq s\right\}$, $\forall Z \in \sigma\left\{X_k : k \geq t+s\right\}$ and $\forall I \in \Re^n$; (iv) $\sum_{t=1}^{\infty}\left\|Cov\left(X_t, X_0\right)\right\| < \infty$ and $\Lambda_0 = \Lambda + \sum_{t=1}^{\infty} Cov\left(X_t, X_0\right)$ is non-singular. Condition (ii) is similar to the Lyapounov condition, and is used to show the uniform asymptotic negligibility condition of Lindeberg for the Central Limit Theorem to hold. The total variability of the sum, S_T, on the other hand, is always required to grow to infinity. Condition (iii) ensures asymptotic independence; it is required for applying the Central Limit Theorem for non-i.i.d. random variables or random vectors. The first part of condition (iv), the finiteness condition, implies that Λ_0 exists and is finite, and that S_T in condition (ii) grows at the same rate as T. Finally the second part – the non-singularity of Λ_0 – is required to get a non-degenerate asymptotic distribution when applying the Central Limit Theorem. All these (weak) conditions are necessary to apply the result 1 in Basak et al. (2002, p. 1203) and identify a distribution for the vector $\overline{X}_T$:

$$\sqrt{T}\left(\overline{X}_T - X\right) \xrightarrow{d} N\left(0, \Lambda_0\right)$$

The second step is to obtain the asymptotic distribution of $\ell\left(e_b, s_b^2, e, S, \gamma\right) = f\left(\overline{X}_T, \gamma\right)$. In order to apply the delta method, the optimal solution λ_b in (3) has to be a smooth function of the parameters. For this to be satisfied, whenever an inequality constraint is binding, the corresponding Lagrange multiplier should be strictly positive. Following our previous notation we need, in other words, the vectors of constraints plus the vectors of Lagrange multipliers δ_2 and δ_3 associated with the inequality constraints to be strictly positive:

$$\left(w^* - lb\right) + \delta_2 > 0$$

$$\left(ub - w^*\right) + \delta_3 > 0$$

The following conditions (v) and (vi) ensure that this is the case. In order to state the assumptions, however, we need the following additional notation. Let $\left\{i_1, \ldots, i_k, i_{k+1}, \ldots i_n\right\}$ be any permutation of

$(1,\ldots,n)$. Let $\left(\Sigma^{-1}\right)_2$ be the $(n-k)\times n$ matrix consisting of the $\{i_{k+1},\ldots i_n\}$ rows of the Σ^{-1}; $\left(\Sigma^{-1}\right)_{22}$ be the $(n-k)\times(n-k)$ principal minor matrix which consists of $\{i_{k+1},\ldots i_n\}$ rows and columns of the Σ^{-1}; Σ_{11}^{-1} be the inverse of the $k\times k$ principal minor matrix corresponding to the $\{i_1,\ldots i_k\}$ rows and columns of the Σ. Lastly, let lb_2 be the vector consisting of the $\{i_{k+1},\ldots i_n\}$ rows of lb, and ub_2 be the corresponding vector of the $\{i_{k+1},\ldots i_n\}$ rows of ub.

We thus modify assumption (7) in Basak et al. (2002) and require that (v) all the elements of the $(n-k)$-dimensional vector $\left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\gamma lb_2-\left(\Sigma^{-1}\right)_2(\eta-A'\delta_1)\right)$ are strictly positive. It turns out that this condition (v) is sufficient to ensure that $\left(w^*-lb\right)+\delta_2$ is a strictly positive vector.

Note indeed that the first order condition to the maximization problem implies that

$$\gamma\Sigma w=\eta-A'\delta_1+\delta_2-\delta_3.$$

Hence

$$w^*=\frac{1}{\gamma}\left(\Sigma^{-1}(\eta-A'\delta_1)+\Sigma^{-1}(\delta_2-\delta_3)\right). \tag{5}$$

To see how assumption (v) works, suppose now that only some elements of w^* given by the subvector $\left[w^*_{i_{k+1}},\ldots,w^*_{i_n}\right]$, are such that the constraint $w^*\geq lb$ holds with equality. We need to show that the corresponding subvector of Lagrange multipliers, $\left(\delta_{2i_{k+1}},\ldots,\delta_{2i_n}\right)$ is positive. Without loss of generality assume that the last $(n-k)$ elements of w^* are such that the lower-inequality constraint is binding. Denote the vector of these elements as w_2^* and the rest as w_1^*, i.e., $w^*=\left[w_1^{*\prime}\quad w_2^{*\prime}\right]'$; therefore $w_2^*=lb_2$. Let δ_{21} and δ_{22}, δ_{31} and δ_{32} denote the corresponding partition of the Lagrange multiplier vector, i.e., $\delta_2=\left[\delta_{21}{}'\quad\delta_{22}{}'\right]'$ and $\delta_3=\left[\delta_{31}{}'\quad\delta_{32}{}'\right]'$. From the constraint $w^*-lb\geq 0$, δ_{21} is a zero vector. Furthermore, $\delta_{32}=0$. Now partition Σ, Σ^{-1}, η and A, similarly, as

$$\Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix};\qquad\qquad \Sigma^{-1}=\begin{bmatrix}\Sigma^{11} & \Sigma^{12}\\ \Sigma^{21} & \Sigma^{22}\end{bmatrix}$$

$\eta = \begin{bmatrix} \eta_1' & \eta_2' \end{bmatrix}'$, and $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, with Σ_{11} and Σ^{11} $k \times k$ matrices, η_1 and $A_1'\delta_1$ are $k \times 1$ vectors. This gives $w_2^* = \frac{1}{\gamma}\left(\begin{bmatrix} \Sigma^{21} & \Sigma^{22} \end{bmatrix}(\eta - A'\delta_1) + \Sigma^{22}\delta_{22}\right) = lb_2$. Thus,

$$\delta_{22} = \left(\Sigma^{22}\right)^{-1}\left(\gamma lb_2 - \begin{bmatrix} \Sigma^{21} & \Sigma^{22} \end{bmatrix}(\eta - A'\delta_1)\right) = \left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\gamma lb_2 - \left(\Sigma^{-1}\right)_2(\eta - A'\delta_1)\right)$$

which is positive by the assumption.

Analogously, to conclude that $\delta_{32} > 0$ when $w_2^* = ub_2$ we need the following assumption (vi): all the elements of the $(n-k)$-dimensional vector $\left(\left(\Sigma^{-1}\right)_{22}\right)^{-1}\left(\left(\Sigma^{-1}\right)_2(\eta - A'\delta_1) - \gamma ub_2\right)$ are strictly positive. If so, $\delta_{32} = \left(\Sigma^{22}\right)^{-1}\left(\begin{bmatrix} \Sigma^{21} & \Sigma^{22} \end{bmatrix}(\eta - A'\delta_1) - \gamma ub_2\right) + \delta_{22} = \left(\Sigma^{22}\right)^{-1}\left(\begin{bmatrix} \Sigma^{21} & \Sigma^{22} \end{bmatrix}(\eta - A'\delta_1) - \gamma ub_2\right) > 0$.

In case conditions (v) and (vi) hold true, therefore, $\ell\left(e_b, s_b^2, e, S, \gamma\right)$ is a continuous function with a continuous first derivative in any point except for its boundaries. By means of the delta method we thus obtain

$$\sqrt{T}\left(\ell\left(e_b, s_b^2, e, S, \gamma\right) - \lambda\left(\eta_b, \sigma_b^2, \eta, \Sigma, \gamma\right)\right) \xrightarrow{d} N(0, V)$$

with $V = \nabla(\gamma)' \Lambda_0 \nabla(\gamma)$, where

$$\nabla(\gamma) = \frac{\partial \lambda\left(\eta_b, \sigma_b^2, \eta, \Sigma, \gamma\right)}{\partial X} = \frac{\partial f(X, \gamma)}{\partial X}.$$

Define $\lambda\left(w, \delta \mid \overline{X}_T\right)$ as the Lagrangian and $\delta = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 \end{bmatrix}$ as the set of Lagrange multipliers:

$$\begin{aligned} \lambda\left(w, \delta \mid \overline{X}_T\right) = \left(w'e - \frac{1}{2}\gamma w'Sw\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) + \\ -\delta_1'(Aw - a) - \delta_2'(l - w) - \delta_3'(w - u) \end{aligned}$$

By making use of the envelope theorem, the gradient $\nabla(\gamma)$ is consistently estimated by

$$D(\gamma) = \frac{\partial f\left(\overline{X}_T, \gamma\right)}{\partial \overline{X}_T} = \left.\frac{\partial \lambda\left(w, \delta \mid \overline{X}_T\right)}{\partial \overline{X}_T}\right|_{\substack{w = w^* \\ \delta = \delta^*}}.$$

The derivative is worth

$$D(\gamma)' = \left[\left(w^* + \left(\gamma e' w^* \right) w^* \right)' \quad -1-\gamma e_b \quad -\frac{1}{2}\gamma \left[w_1^{*2} \quad 2w_1^* w_2^* \quad \cdots \quad 2w_1^* w_n^* \quad w_2^{*2} \quad \cdots \quad 2w_2^* w_n^* \quad \cdots \quad w_n^{*2} \right] \quad \frac{1}{2}\gamma \right]$$

Lastly, we replace Λ_0 with its standard heteroskedasticity and autocorrelation consistent estimate L_0 as proposed by Newey and West (1987) and make use of Bartlett-type weights:

$$L_0 = \hat{\Omega}_0 + \sum_{j=1}^{m} \left(1 - \frac{j}{m+1} \right) (\hat{\Omega}_j + \hat{\Omega}_j{}')$$

with

$$\hat{\Omega}_j = \frac{1}{T} \sum_{t=j+1}^{T} \left(X_t - \overline{X}_T \right) \left(X_{t-j} - \overline{X}_T \right)'$$

and m the number of lags to be considered. As suggested by Newey and West (1994), good asymptotic properties can be achieved by using the automatic lag selection rule

$$m = \operatorname{int}\left(4 \left(\frac{T}{100} \right)^{\frac{2}{9}} \right).$$

Consider therefore the statistic

$$t = T^{1/2} \frac{\ell_b - \lambda_b}{\hat{V}^{1/2}} = T^{1/2} \frac{\ell_b - \lambda_b}{\left(D(\gamma)' L_0 D(\gamma) \right)^{1/2}}$$

Under the null hypothesis $H_0 : \lambda_b = \lambda_0$, $t \overset{a}{\sim} N(0,1)$. Notice that the null can be equivalently written as $H_0 : cv_b = cv_0 = 1 - \exp\{-\lambda_0\}$. This second specification highlights a shortcoming of this procedure: since $cv_b \in (-\infty, 1]$, we are not able to test whether $cv_b = 1$. A similar issue arises in Snedecor and Cochran (1989), when trying to test a null hypothesis on a variance $\sigma^2 = 0$. In their framework, a statistic with an exact distribution exists for any value of the variance, except for $\sigma^2 = 0$, i.e., on the boundary of the feasible set. An analogous situation is reported in Kim et al. (2005) when dealing with Sharpe-style regressions, used to investigate issues such as style composition, style sensitivity and style change over time. The method employed to obtain the distribution and confidence intervals of the style coefficients are statistically valid only when none of the true style weights are zero or one. In practice, it seems to be quite plausible to have zero or one as the values of some style weights. In our framework, nevertheless, such a hypothesis is not economically relevant: it is, indeed, hard to imagine a benchmark, however badly managed, able to dissipate *all* the wealth. We can, however, test any other hypothesis, and in particular if $cv_b = 0$, that is, if the benchmark can perfectly replicate the performance of the optimal portfolio.

Since we know the large sample distribution for $\ell\left(e_b, s_b^2, e, S, \gamma\right)$, a confidence interval for λ_b is derived by

$$\alpha = P\left(z_{\frac{\alpha}{2}} \leq \sqrt{T}\,\frac{\ell_b - \lambda_0}{\sqrt{\hat{V}}} \leq z_{1-\frac{\alpha}{2}} \right) =$$

$$= P\left(\ell_b - z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{T}\hat{V}} \leq \lambda_0 \leq \ell_b + z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{T}\hat{V}} \right)$$

where $z_{1-\frac{\alpha}{2}}$ is the $1-\dfrac{\alpha}{2}$-eth percentile of a standard normal distribution.

Since $cv_0 = 1-\exp\{-\lambda_0\}$, a confidence interval for the wealth loss is

$$CI\left(cv_0\right) = \left\{ cv_b : cv_b \in \left[1-\exp\left\{-\left(\ell_b - z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{T}\hat{V}}\right)\right\}, 1-\exp\left\{-\left(\ell_b + z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{T}\hat{V}}\right)\right\}\right]\right\}.$$

If we are interested in testing portfolio efficiency, once we define

$$\overline{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t = \begin{bmatrix} e \\ vech\left(M\right) \end{bmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} e_t \\ vech\left(M_t\right) \end{bmatrix}$$

it is straightforward to see that

$$D(\gamma)' = \left[\left(w^* + \left(\gamma e' w^*\right) w^*\right)' \quad -\frac{1}{2}\gamma\left[w_1^{*2} \quad 2w_1^* w_2^* \quad \cdots \quad 2w_1^* w_n^* \quad w_2^{*2} \quad \cdots \quad 2w_2^* w_n^* \quad \cdots \quad w_n^{*2} \right] \right] +$$
$$-\left[\left(\omega + \left(\gamma e'\omega\right)\omega\right)' \quad -\frac{1}{2}\gamma\left[\omega_1^2 \quad 2\omega_1\omega_2 \quad \cdots \quad 2\omega_1\omega_n \quad \omega_2^2 \quad \cdots \quad 2\omega_2\omega_n \quad \cdots \quad \omega_n^2 \right] \right]$$

and that a confidence interval for the wealth loss cv_0 is

$$CI\left(cv_0\right) = \left\{ cv_p : cv_p \in \left[1-\exp\left\{-\left(\ell_p - z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{T}\hat{V}}\right)\right\}, 1-\exp\left\{-\left(\ell_p + z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{T}\hat{V}}\right)\right\}\right]\right\}$$

This specification of the test does not hold true for cv_p equal to 0 or 1; in this context, therefore, we are not allowed to test either $H_0 : cv_0 = 1$ or $H_0 : cv_0 = 0$. In particular, we cannot test whether the observed portfolio is efficient or not. As in Snedecor and Cochran (1989), however, we may rely on the confidence interval to cv_0 and check how far its lower (upper) boundary is from zero (one).

4. Closed-form solutions with no inequality constraints

The expression of the test derived in §3 still depends on the optimal portfolios. We are able to establish their closed-form expression only in the simplest settings, with no inequality constraints; otherwise we have to rely on numerical solutions. For instance, a Matlab® code which implements the function `quadprog` can solve the problem numerically.

The closed-form solution is feasible when i) there are no constraints at all or ii) there are only equality constraints. Below we consider the two cases separately. We establish, moreover, that a strong relationship between standard mean-variance and utility paradigms exists; the link is provided by deriving the optimal portfolios. In the next part we show the results taking into account only the benchmark case; analogous results apply in the portfolio framework.

4.1. No constraints

We call $\ell^{NO}\left(e_b, s_b^2, e, S, \gamma\right)$ the difference between utilities in the case of no constraints:

$$\ell^{NO}\left(e_b, s_b^2, e, S, \gamma\right) = \max_w \left\{\left(w'e - \frac{1}{2}\gamma w'Sw\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right)\right\}$$

Deriving $\ell^{NO}\left(e_b, s_b^2, e, S, \gamma\right)$ with respect to w we get:

$$\frac{\partial \ell^{NO}\left(e_b, s_b^2, e, S, \gamma\right)}{\partial w} = e - \gamma S w = 0$$

so that the optimal weights are

$$w_{NO}^* = \frac{1}{\gamma} S^{-1} e \tag{6}$$

Replacing the expression (6) into $\ell^{NO}\left(e_b, s_b^2, e, S, \gamma\right)$ we have then

$$\begin{aligned}
\ell^{NO}\left(e_b, s_b^2, e, S, \gamma\right) &= \left(w_{NO}^{*\prime} e - \frac{1}{2}\gamma w_{NO}^{*\prime} S w_{NO}^*\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) = \\
&= \left(\frac{1}{\gamma} e' S^{-1} e - \frac{1}{2\gamma} e' S^{-1} e\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) = \\
&= \frac{1}{2\gamma} e' S^{-1} e - \left(e_b - \frac{1}{2}\gamma s_b^2\right)
\end{aligned}$$

We recognize in equation (6) an expression similar to that in the standard mean-variance analysis with no restrictions, where the optimal portfolio can be any of the infinite ones with the highest Sharpe ratio. The weights of the optimal portfolio with the same excess return r_b as the benchmark are then given by

(7) $$w_{NO}^{BJS} = \frac{r_b S^{-1} e}{e' S^{-1} e}$$

This expression identifies the optimal portfolio used in Basak et al. (2002), where an agent aims at minimizing the variance of her investment given the expected return r_b.

It can be shown that, when we impose that the portfolio weights sum to one then:

- the Sharpe ratio of the portfolio (7) is equivalent to the Sharpe ratio of the tangency portfolio (TP),

$$w_{TP}^{BJS} = \frac{S^{-1} e}{\iota' S^{-1} e}$$

- the optimal portfolio that maximizes the expected utility of the agent with the highest optimal expected utility is the tangency portfolio, i.e., exactly the same portfolio we have in a standard mean-variance setting.

The optimal portfolio resulting in our expected utility framework in the case of no constraint is: i) equivalent to the optimal one in the mean-variance framework if there are no risk free assets and ii) is otherwise proportional. Indeed, both equations (6) and (7) share the same numerator $S^{-1}e$; the different denominators just normalize the weights. In other words, the importance of the two quantities

$$\frac{e' S^{-1} e}{r_b}; \qquad \gamma$$

is in defining what fraction of wealth, if any, should be invested in the risky assets and consequently in the risk free; the relationship between risky shares is instead kept fixed. This implies that the two portfolios are on the same efficient frontier; see for instance the two optimal portfolios in figure 1. According to the two fund separation theorem, they could be seen as a combination of the tangency risky portfolio and a risk free asset.

4.2. Equality constraints only

If, instead, we define the function $\ell^{EQ}\left(e_b, s_b^2, e, S, \gamma\right)$ that takes account of equality constraints on some of the optimal portfolio weights,

$$\ell^{EQ}\left(e_b, s_b^2, e, S, \gamma\right) = \max_w \left\{ \left(w'e - \frac{1}{2}\gamma w' S w \right) - \left(e_b - \frac{1}{2}\gamma s_b^2 \right) \right\}$$

subject to

$$Aw = a$$

the Lagrangian is

$$\lambda\left(w,\delta_1 \mid \overline{X}_T\right) = w'e - \frac{1}{2}\gamma w'S - we_b + \frac{1}{2}\gamma s_b^2 - \delta_1'\left(Aw - a\right)$$

If we take the derivative with respect to w,

$$\frac{\partial \lambda\left(w,\delta_1 \mid \overline{X}_T\right)}{\partial w} = 0$$

(8)
$$\Rightarrow -e + \gamma Sw - A'\delta_1 = 0$$

$$\Rightarrow w = \frac{1}{\gamma}S^{-1}\left(e + A'\delta_1\right)$$

and with respect to δ_1,

$$\frac{\partial \lambda\left(w,\delta_1 \mid \overline{X}_T\right)}{\partial \delta_1} = 0$$

$$\Rightarrow Aw = a$$

we face a system of two equations that can be solved premultiplying (8) by A,

$$Aw = a = \frac{1}{\gamma}AS^{-1}\left(e + A'\delta_1\right)$$

$$\Rightarrow \delta_1^* = \left(AS^{-1}A'\right)^{-1}\left(\gamma a - AS^{-1}e\right)$$

from which

$$w_{EQ}^* = \frac{1}{\gamma}S^{-1}\left(e + A'\delta_1^*\right) =$$

$$= \frac{1}{\gamma}\left(I - S^{-1}A'\left(AS^{-1}A'\right)^{-1}A\right)S^{-1}e + S^{-1}A'\left(AS^{-1}A'\right)^{-1}a = \frac{1}{\gamma}Q + q$$

Replacing this expression in the objective function we have

$$\ell^{EQ}\left(e_b, s_b^2, e, S, \gamma\right) = \left(w_1^{*\prime}e - \frac{1}{2}\gamma w_1^{*\prime}Sw_1^*\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) =$$

$$= \left(\frac{1}{\gamma}Q'e + q'e - \frac{1}{2}\gamma\left(\frac{1}{\gamma^2}Q'SQ + \frac{1}{\gamma}Q'Sq + \frac{1}{\gamma}q'SQ + q'Sq\right)\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right) =$$

$$= \left(\frac{1}{\gamma}Q'e + q'e - \frac{1}{2\gamma}Q'SQ - \frac{1}{2}Q'Sq - \frac{1}{2}q'SQ - \frac{1}{2}\gamma q'Sq\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right)$$

In order to make a comparison with the existing literature, it turns helpful to split the primitive assets in two groups[7]:

$$w = \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix}; \qquad e = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}; \qquad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix}$$

and to deal with the constraint

$$\tilde{w}_2 = \tilde{\omega}_2 .$$

After some algebra we obtain

$$\tilde{w}_1^* = \frac{1}{\gamma} S_{11}^{-1} \tilde{e}_1 - S_{11}^{-1} S_{12} \tilde{\omega}_2 \tag{9}$$

In selecting the optimal values, an agent has then to take into account a hedge term against the constrained assets. It is interesting to deal with an equality constraint because it allows us to model the presence of transaction costs in some assets that, for this reason, are illiquid. For instance, using Italian data and the Gourieroux and Jouneau (GJ, 1999) test, Pelizzon and Weber (2003) observe that housing is an important part (nearly 80%) of the overall wealth of Italian households, and the efficiency greatly improves when real assets are taken as a fixed component of the overall portfolio. Bucciol (2003) bears out their results and shows that the efficiency improves further when inequality constraints are also taken into account.

In a setting à la Gourieroux and Jouneau (1999), we would be given the optimal portfolio as

$$w_{EQ}^{GJ} = w_{EQ}^{BJS} = \begin{cases} \dfrac{\left(r_b - \tilde{\omega}_2' \left(\tilde{e}_2 - S_{12}' S_{11}^{-1} \tilde{e}_1\right)\right)}{\tilde{e}_1' S_{11}^{-1} \tilde{e}_1} S_{11}^{-1} \tilde{e}_1 - S_{11}^{-1} S_{12} \tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

where r_b is the expected excess return on the observed portfolio. Given the expected return r_b the optimal portfolio is exactly the same when computed with the test of Basak et al. (2002).

Moreover, with the restriction on the sum of weights

$$w_{TP}^{BJS} = \begin{cases} \left(\dfrac{1 - \iota' \tilde{\omega}_2 + \iota' S_{11}^{-1} S_{12} \tilde{\omega}_2}{\iota' S_{11}^{-1} \tilde{e}_1} \right) S_{11}^{-1} \tilde{e}_1 - S_{11}^{-1} S_{12} \tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

In our utility framework, instead, extending equation (9) to all the primitive assets, the optimal portfolio is given by

[7] This setting was used in Gourieroux and Jouneau (1999). Their statistic stems from a restricted mean-variance space, where the unconstrained portfolio shares are normalized by the constrained shares.

$$w_{EQ}^{*} = \begin{cases} \dfrac{1}{\gamma} S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1} S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

and, in the case we require the sum to one,

$$w_{EQ}^{**} = \begin{cases} \left(\dfrac{1 - \iota'\tilde{\omega}_2 + \iota' S_{11}^{-1} S_{12}\tilde{\omega}_2}{\iota' S_{11}^{-1}\tilde{e}_1} \right) S_{11}^{-1}\tilde{e}_1 - S_{11}^{-1} S_{12}\tilde{\omega}_2 \\ \tilde{\omega}_2 \end{cases}$$

i.e., exactly the same equation obtained in a setting à la Basak et al. (2002). Without imposing the sum to one, the only difference with GJ and BJS tests is, as before, in the normalization term: on the one hand, we have the expression

$$\frac{\left(r_b - \tilde{\omega}_2' \left(\tilde{e}_2 - S_{12}' S_{11}^{-1}\tilde{e}_1 \right) \right)}{\tilde{e}_1' S_{11}^{-1}\tilde{e}_1}$$

whereas, on the other, we have only the term γ. The same remarks made in §4.1 apply here.

In summary, despite slight differences the behavior in a setting with no inequality constraints is similar to the mean-variance framework. If we add inequality constraints, instead, we do not have any closed-form solution for the optimal portfolios, and therefore we are not able to make any analytical comparison.

5. The relative risk aversion parameter

The knowledge of the relative risk aversion parameter γ is critical to asset allocation choice since it is decisive in determining the level of investment in risky assets, as we see for example in equation (6).

By definition, γ depends neither on time nor wealth:

$$\gamma = -W_t \frac{U''(W_t)}{U'(W_t)}$$

It is well known, however, (see Stutzer, 2004, for a review) that its exact value for an investor is as hard to know as it is to estimate it through an *ad hoc* question. Rabin and Thaler (2001) believe that any method used to measure a coefficient of relative risk aversion is doomed to failure, since *«the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of relative risk aversion, so it is a waste of time to try to measure it».*

In this section we show that it is possible to provide an estimate of the relative risk aversion parameter γ within this framework. Our procedure is closely related to that in Gourieroux and Monfort (2005); they test their hypothesis using a statistic which depends on an exogenous preference parameter. Should the parameter not *a priori* be given, they obtain an estimate by minimizing the statistic with respect to such a parameter. In our setting, the role of the preference parameter is played by γ, the risk aversion coefficient. By solving a similar problem for the objective function we can empirically find the implied risk aversion parameter, the one for which the welfare loss is minimized. Under the hypothesis that the portfolio is managed in order to maximize the expected utility function, the estimator $\hat{\gamma}$ then provides a consistent estimate for the utility function.

It is straightforward to develop a procedure for deriving $\hat{\gamma}$ in a portfolio setting. Since the function $\ell(e,S,\gamma)$ is always non-negative, we can estimate γ by choosing the value that makes the objective function as small as possible, i.e., leads to the lowest inefficiency. In formulae, we solve

$$\hat{\gamma} = \arg\min_{\gamma} \max_{w} \left\{ \left(w'e - \frac{1}{2}\gamma w'Sw \right) - \left(\omega'e - \frac{1}{2}\gamma\omega'S\omega \right) \right\}$$

subject to several constraints.

5.1. No constraints

If there are no restrictions, the optimal γ is chosen by

$$\hat{\gamma}_{NO} = \arg\min_{\gamma} \left\{ \frac{1}{2\gamma} e'S^{-1}e + \frac{1}{2}\gamma\omega'S\omega - \omega'e \right\}$$

It leads us to the first order condition

$$\frac{1}{2}\omega'S\omega - \frac{1}{2\gamma^2} e'S^{-1}e = 0$$

which implies

$$\hat{\gamma}_{NO} = \sqrt{\frac{e'S^{-1}e}{\omega'S\omega}} \tag{10}$$

Knowing its analytical expression, we can also derive a standard error and a confidence interval for $\hat{\gamma}_{NO}$, making use of the same results in §3.

Let us define

$$\overline{Y}_T = \begin{bmatrix} e \\ vech(S) \end{bmatrix} = g\left(\begin{bmatrix} e \\ vech(S+ee') \end{bmatrix}\right) = g\left(\overline{X}_T\right)$$

and $\hat{\gamma}_{NO} = h\left(\overline{Y}_T\right)$ with

$$V\left(\overline{X}_T\right) = \frac{\partial g\left(\overline{X}_T\right)}{\partial \overline{X}_T} = \begin{bmatrix} I_n & \underset{n\times\frac{n(n+1)}{2}}{\mathbf{0}} \\ \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix} & I_{\frac{n(n+1)}{2}} \end{bmatrix}$$

and

$$G_i = -\begin{bmatrix} \underset{(n-i+1)\times(i-1)}{\mathbf{0}} & \begin{bmatrix} e_i \\ e_{i+1} \\ \vdots \\ e_n \end{bmatrix} & \underset{(n-i+1)\times(n-i)}{\mathbf{0}} \end{bmatrix} - e_i \begin{bmatrix} \underset{(n-i+1)\times(i-1)}{\mathbf{0}} & I_{n-i+1} \end{bmatrix}$$

Moreover,

$$Z\left(\overline{Y}_T\right)' = \left(\frac{\partial h\left(\overline{Y}_T\right)}{\partial \overline{Y}_T}\right)' =$$

$$= \begin{bmatrix} \underset{1\times n}{\mathbf{0}} & \frac{1}{2\sqrt{\omega' S\omega}\sqrt{e'S^{-1}e}}\begin{bmatrix}\omega_1^2 & 2\omega_1\omega_2 & \cdots & 2\omega_1\omega_n & \omega_2^2 & \cdots & 2\omega_2\omega_n & \cdots & \omega_n^2\end{bmatrix}\end{bmatrix} +$$

$$-\begin{bmatrix} \frac{e'S^{-1}}{\sqrt{\omega' S\omega}\sqrt{e'S^{-1}e}} & \frac{\sqrt{e'S^{-1}e}}{2\left(\omega' S\omega\right)^{3/2}}\left[e'DS_{11}e \quad e'DS_{12}e \quad \cdots \quad e'DS_{1n}e \quad e'DS_{22}e \quad \cdots \quad e'DS_{23}e \quad \cdots \quad e'DS_{nn}e\right]\end{bmatrix}$$

where DS_{ij} denotes the derivative of the (i, j)-eth element of S^{-1}.

Therefore, the standard error for $\hat{\gamma}_{NO}$ is

$$s.e.\left(\hat{\gamma}_{NO}\right) = \sqrt{\frac{1}{T} Z\left(\overline{Y}_T\right)' V\left(\overline{X}_T\right) L_0 V\left(\overline{X}_T\right)' Z\left(\overline{Y}_T\right)}$$

and, applying the central limit theorem, a confidence interval for $\hat{\gamma}_{NO}$ is

$$CI\left(\hat{\gamma}_{NO}\right) = \left\{\gamma > 0 : \gamma \in \left[\hat{\gamma}_{NO} - z_{1-\frac{\alpha}{2}} s.e.\left(\hat{\gamma}_{NO}\right), \hat{\gamma}_{NO} + z_{1-\frac{\alpha}{2}} s.e.\left(\hat{\gamma}_{NO}\right)\right]\right\}$$

5.2. Other constraints

Analogously, in the case of equality constraints only, it is necessary to solve

$$\hat{\gamma}_{EQ} = \arg\min_{\gamma} \left\{ \left(\frac{1}{\gamma} Q'e + q'e - \frac{1}{2\gamma} Q'SQ - \frac{1}{2} Q'Sq - \frac{1}{2} q'SQ - \frac{1}{2} \gamma q'Sq \right) - \left(\omega' e - \frac{1}{2} \gamma \omega' S \omega \right) \right\}$$

Deriving with respect to γ,

$$\frac{1}{2} \omega' S \omega - \frac{1}{\gamma^2} Q'e + \frac{1}{2\gamma^2} Q'SQ - \frac{1}{2} q'Sq = 0$$

(11)
$$\hat{\gamma}_{EQ} = \sqrt{\frac{2Q'e - Q'SQ}{\omega' S \omega - q'Sq}}$$

although it is not always true that a real solution exists (it should be $\omega' S \omega < q'Sq$)

When inequality constraints are also present, it is no longer possible to find an exact expression for the estimate of the risk aversion parameter γ; we know, nevertheless, that the function

$$\min_{\gamma} \ell(e, S, \gamma) = \min_{\gamma} \max_{w} \left\{ \left(w'e - \frac{1}{2} \gamma w' S w \right) - \left(\omega' e - \frac{1}{2} \gamma \omega' S \omega \right) \right\}$$

determines the first order condition

$$\frac{\partial \ell(e, S, \gamma)}{\partial \gamma} = \frac{\partial \lambda(e, S, \delta)}{\partial \gamma} \Big| w = w^* = \frac{1}{2} \omega' S \omega - \frac{1}{2} w^*(\gamma)' S w^*(\gamma) = 0$$

The optimal γ, therefore, is implicitly defined by the equation

$$\omega' S \omega = w^*(\gamma)' S w^*(\gamma)$$

The same argument does not hold for the benchmark case, where the function $\ell\left(e_b, s_b^2, e, S, \gamma\right)$ can indeed take on both positive and negative values. In this case we might consider either the value that maximizes the objective function (i.e., the benchmark gets the highest efficiency), or the value that makes the objective function null (i.e., the benchmark is as efficient as the optimal portfolio).

When we consider the value of γ that maximizes the objective function, we simply need to adjust the formulae already derived in the portfolio context:

$$\hat{\gamma}_{NO} = \sqrt{\frac{e' S^{-1} e}{s_b^2}} \; ; \qquad \hat{\gamma}_{EQ} = \sqrt{\frac{2Q'e - Q'SQ}{s_b^2 - q'Sq}}$$

In the presence of inequality constraints, the same situation arises.

For $\hat{\gamma}_{NO}$ we can also define the standard error, corresponding to the one obtained in a portfolio setting. What changes is that

$$\overline{Y}_T = \begin{bmatrix} e \\ e_b \\ vech(S) \\ s_b^2 \end{bmatrix} = g\left(\overline{X}_T\right)$$

and $\hat{\gamma}_{NO} = h\left(\overline{Y}_T\right)$ with

$$V\left(\overline{X}_T\right) = \frac{\partial g\left(\overline{X}_T\right)}{\partial \overline{X}_T} = \begin{bmatrix} I_n & \underset{n\times 1}{\mathbf{0}} & \underset{n\times\frac{n(n+1)}{2}}{\mathbf{0}} & \underset{n\times 1}{\mathbf{0}} \\ \underset{1\times n}{\mathbf{0}} & 1 & \underset{1\times\frac{n(n+1)}{2}}{\mathbf{0}} & 0 \\ \begin{bmatrix} G_1 \\ \vdots \\ G_k \end{bmatrix} & \underset{\frac{n(n+1)}{2}\times 1}{\mathbf{0}} & I_{\frac{n(n+1)}{2}} & \underset{\frac{n(n+1)}{2}\times 1}{\mathbf{0}} \\ \underset{1\times n}{\mathbf{0}} & -2e_b & \underset{1\times\frac{n(n+1)}{2}}{\mathbf{0}} & 1 \end{bmatrix}$$

Finally,

$$Z\left(\overline{Y}_T\right) = \frac{\partial h\left(\overline{Y}_T\right)}{\partial \overline{Y}_T} = \begin{bmatrix} \dfrac{S^{-1}e}{\sqrt{s_b^2}\sqrt{e'S^{-1}e}} \\ 0 \\ \dfrac{1}{2\sqrt{e'S^{-1}e}\sqrt{s_b^2}} \begin{bmatrix} e'DS_{11}e \\ e'DS_{12}e \\ \vdots \\ e'DS_{1n}e \\ e'DS_{22}e \\ e'DS_{23}e \\ \vdots \\ e'DS_{nn}e \end{bmatrix} \\ -\dfrac{\sqrt{e'S^{-1}e}}{2\left(s_b^2\right)^{3/2}} \end{bmatrix}$$

6. Empirical analysis

We perform two separate empirical analyses on the efficiency of a benchmark and of a portfolio. As a benchmark we use the S&P 500 index[8] against a set of ten industry portfolios

[8] Downloaded from http://www.yahoo.com.

representative of the U.S. market[9]. The industry is divided into non-durable, durable, manufacturing, energy, hi-tech, telecommunication, shops, health, utilities and other sectors. We consider monthly returns that cover the period February 1950 through May 2005 (664 observations).

Table 1 reports some descriptive statistics for our sample; we observe from panel A that the expected return of the benchmark is lower than that of any other primitive asset. This fact has a critical impact on obtaining the optimal portfolio when several constraints are required. In a Basak et al. (2002) framework, for instance, the efficient portfolio must have the same mean as the benchmark. If using these data we also impose short-sale constraints, the problem cannot be solved, since it is not possible to obtain any portfolio with such a low mean.

In panel B we notice, moreover, that the utilities industry sector guarantees a lower variance than the benchmark. This asset therefore *dominates* the benchmark. We consequently expect the benchmark to be an inefficient financial instrument and that our test will detect a high wealth loss.

Table 1.

Descriptive statistics for industry portfolios and benchmark returns

Panel A: Mean

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Benchmark
	1.0836	1.0241	1.0144	1.1960	1.1993	0.9088	1.0342	1.1782	0.9312	1.0836	0.7274

Panel B: Covariance (normal) and correlation (italic) of percentage returns

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Benchmark
NoDur	17.892	*0.64166*	*0.81769*	*0.49454*	*0.57554*	*0.62724*	*0.83830*	*0.74980*	*0.63453*	*0.82366*	*0.82558*
Durbl	14.968	30.414	*0.78544*	*0.46413*	*0.62017*	*0.57020*	*0.74695*	*0.49183*	*0.45877*	*0.75108*	*0.78984*
Manuf	16.379	20.512	22.425	*0.62428*	*0.74151*	*0.61882*	*0.82430*	*0.72566*	*0.54838*	*0.89333*	*0.91494*
Enrgy	10.578	12.943	14.948	25.569	*0.41925*	*0.39057*	*0.44976*	*0.44836*	*0.54592*	*0.60415*	*0.68284*
HiTec	15.791	22.185	22.777	13.751	42.075	*0.59744*	*0.6874*	*0.63587*	*0.31595*	*0.71070*	*0.80680*
Telcm	11.334	13.434	12.518	8.4369	16.555	18.250	*0.6568*	*0.54124*	*0.53258*	*0.67322*	*0.74720*
Shops	17.389	20.201	19.142	11.153	21.866	13.759	24.049	*0.66010*	*0.51009*	*0.84001*	*0.84336*
Hlth	15.757	13.475	17.072	11.263	20.491	11.487	16.082	24.681	*0.47911*	*0.71912*	*0.76917*
Utils	10.262	9.6734	9.9285	10.554	7.8355	8.6987	9.5639	9.1005	14.618	*0.61203*	*0.60763*
Other	17.037	20.256	20.687	14.939	22.543	14.064	20.144	17.471	11.443	23.914	*0.91609*
Benchmark	14.430	18.000	17.904	14.268	21.625	13.190	17.090	15.790	9.600	18.512	17.076

Using these data, we compute the optimal portfolios for our t test with different levels of risk aversion, imposing different constraints (nothing, non-negativity constraints, equality constraint on one asset, both kinds of constraints). The equality constraint in the residual industry sector is equal to 10%. We impose it after noting that, in the presence of non-negativity constraints, the optimal share of investment in it is zero, and negative in most of the other cases.

[9] Average value-weighted returns, taken from Kenneth French's website: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

In table 2 we report the optimal portfolios for different objective functions and different constraints. For each portfolio it is necessary for the weights to sum to one, i.e., there is no risk free asset. Therefore, when we refer to the unconstrained case, we actually mean that one equality constraint (the sum to one of the weights) holds. Without inequality constraints, the optimal portfolios hold several short positions (1 to 3, according to the level of γ). Such portfolios provide the best performance, but are typically unfeasible in reality, and to compare them with an observed benchmark or an observed portfolio would be misleading. By imposing non-negativity constraints, the optimal portfolios turn out to be composed of only a subset of assets; four primitive assets in particular (durable, manufacturing, shops, other sectors) are never in the investment decisions. Not surprisingly, these are the assets which offer the lowest return/risk profiles, or that correlate highly with other assets.

Table 2.

Optimal portfolios under different risk aversion parameters and subject to different constraints

%	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
No constraints										
μ–Σ	48.5868	14.8428	-43.7188	35.5916	12.9201	13.0905	-3.4854	23.8141	25.7680	-27.4096
BJS*	-30.3313	-6.1700	89.4892	-18.7369	-22.8688	63.9575	19.3286	-8.6478	79.3282	-65.3488
γ=1	267.2189	73.0559	-412.7541	186.1017	112.0687	-127.8298	-66.6888	113.7455	-122.6137	77.6959
γ=2	144.1036	40.2751	-204.9444	101.3470	56.2365	-48.4753	-31.0980	63.1037	-39.0576	18.5093
γ=5	70.2345	20.6067	-80.2585	50.4942	22.7372	-0.8626	-9.7435	32.7186	11.0761	-17.0027
γ=10	45.6114	14.0505	-38.6966	33.5433	11.5708	15.0083	-2.6253	22.5902	27.7873	-28.8400
γ=20	33.2999	10.7725	-17.9156	25.0678	5.9876	22.9438	0.9338	17.5260	36.1429	-34.7586
Non-negativity constraints										
γ=1	0	0	0	52.1807	10.9537	0	0	36.8656	0	0
γ=2	0	0	0	49.8126	7.4591	0	0	42.7282	0	0
γ=5	23.4332	0	0	33.9227	3.5890	0.0102	0	24.2651	14.7799	0
γ=10	17.0364	0	0	23.0100	0	14.6349	0	16.4329	28.8858	0
γ=20	13.5149	0	0	17.1718	0	20.9590	0	11.6415	36.7128	0
Equality constraints (Other = 10%)										
BJS*	-42.8368	-12.1254	69.0879	-27.6070	-29.8115	64.8275	5.3579	-15.3520	78.4594	10
γ=1	272.3957	76.6993	-385.6845	189.8906	115.5132	-124.9638	-53.0151	117.2208	-118.0562	10
γ=2	144.7544	40.7331	-201.5418	101.8233	56.6695	-48.1151	-29.3792	63.5405	-38.4847	10
γ=5	68.1695	19.1534	-91.0561	48.9829	21.3633	-2.0058	-15.1977	31.3323	9.2582	10
γ=10	42.6412	11.9602	-54.2276	31.3694	9.5945	13.3640	-10.4705	20.5963	25.1725	10
γ=20	29.8771	8.3636	-35.8133	22.5627	3.7102	21.0488	-8.1069	15.2282	33.1296	10
Non-negativity and Equality constraints (Other = 10%)										
γ=1	0	0	0	48.9160	8.2057	0	0	32.8784	0	10
γ=2	0	0	0	46.5478	4.7111	0	0	38.7410	0	10
γ=5	18.4565	0	0	32.4582	1.2837	0	0	23.8123	13.9892	10
γ=10	11.9233	0	0	21.0891	0	12.6862	0	14.9635	29.3378	10
γ=20	8.4018	0	0	15.2509	0	19.0103	0	10.1722	37.1648	10

* Optimal portfolio in a mean-variance setting with the same expected return as the benchmark.

In table 3 we summarize the first two moments of returns on optimal portfolios. We observe that, once γ increases, the expected return and the standard deviation of optimal portfolios in a t test setting decrease, but in such a way that the Sharpe ratio grows. On the other hand, the Sharpe ratio for the optimal portfolio in a BJS setting is always much lower, meaning that, when fixing the level of expected utility, we neglect important information for optimal portfolio choice.

Notice, moreover, that there is little difference in performance when an equality constraint is added. When inserting non-negativity constraints, instead, the portfolio shares are completely different, so is their performance. The Sharpe ratio for an optimal portfolio under t test is lower with more constraints: our test, nevertheless, compares utility levels. In the same table we then provide a numerical value for the utility loss, where it is computed as

$$\ell\left(e_b, s_b^2, e, S, \gamma\right) = \left(w^{*\prime} e - \frac{1}{2}\gamma w^{*\prime} S w^{*}\right) - \left(e_b - \frac{1}{2}\gamma s_b^2\right)$$

The utility loss indeed decreases when we add more constraints. Since the constrained optimal portfolio is less efficient than the unconstrained optimal portfolio, the benchmark follows more closely the performance of the best alternative, hence the utility loss is lower (in absolute values) in the presence of constraints.

Table 3.

First two moments of the optimal portfolio returns

%	Mean	Std. Dev.	Sharpe	Utility Loss	Mean	Std. Dev.	Sharpe	Utility Loss
	No constraints				Non-negativity constraints			
μ–Σ	1.1221	3.5463	0.3164	-	-	-	-	-
BJS	0.7278	4.0471	0.1797	-	-	-	-	-
γ=1	2.2155	11.5833	0.1913	0.9025	1.1898	4.2841	0.2777	0.4559
γ=2	1.5998	6.4665	0.2474	0.6247	1.1886	4.2638	0.2788	0.4499
γ=5	1.2303	3.9944	0.3080	0.5303	1.1263	3.8108	0.2956	0.4621
γ=10	1.1072	3.5016	0.3162	0.6193	1.0554	3.5204	0.2998	0.5608
γ=20	1.0456	3.3671	0.31054	0.8895	1.0213	3.4470	0.2963	0.8107
	Equality constraints (Other=10%)				Non-negativity and equality constraints (Other=10%)			
BJS	0.7274	4.2953	0.1693	-	-	-	-	-
γ=1	2.1874	11.4090	0.1917	0.8945	1.1792	4.2537	0.2772	0.4466
γ=2	1.5962	6.4411	0.2478	0.6245	1.1780	4.2332	0.2783	0.4419
γ=5	1.2415	4.0815	0.3042	0.5239	1.1228	3.8538	0.2913	0.4503
γ=10	1.1233	3.6209	0.3102	0.5928	1.0545	3.5790	0.2946	0.5392
γ=20	1.0642	3.4963	0.3044	0.8193	1.0205	3.5069	0.2910	0.7683

Note: the benchmark has a mean of 0.72738, a standard deviation of 4.1292 and a Sharpe ratio of 0.17616.

In figure 3 we plot the optimal portfolios for the t test and their indifference curves against the benchmark; figure 4 shows the same plots for only $\gamma = 5$ and with the efficient frontier. Our test makes a comparison between the indifference curves of the benchmark and the optimal portfolio.

Figure 3.

Efficient portfolios in a mean-standard deviation plan

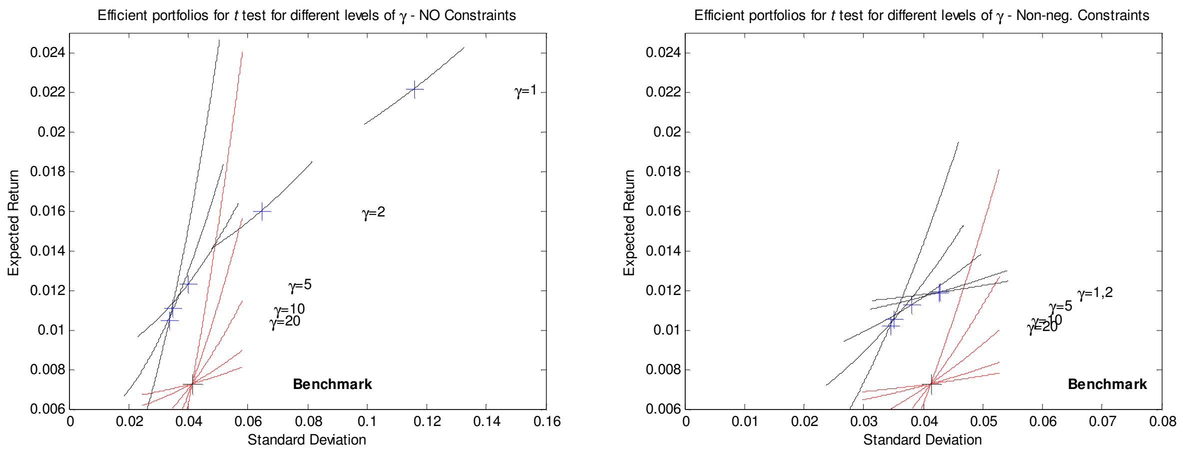
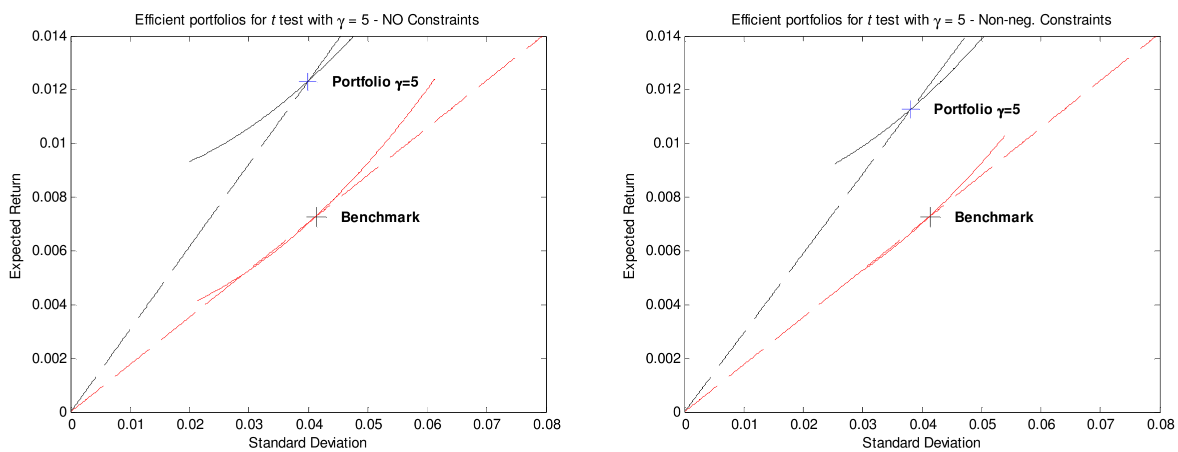


Figure 4.

Efficient portfolios in a mean-standard deviation plan, case γ=5.



6.1. Benchmark case

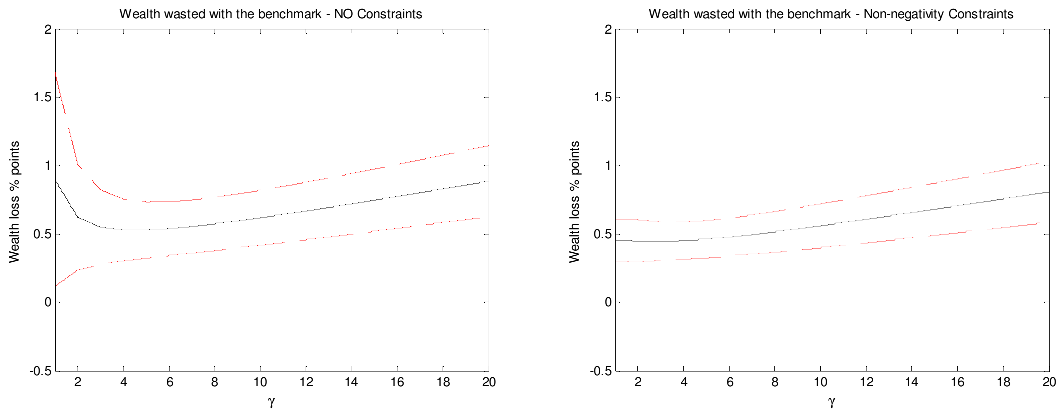
We already know that, by construction, the benchmark is suboptimal in a mean-variance metric[10]. Its inefficiency decreases once we add more constraints; in particular, it decreases appreciably when we impose non-negativity constraints. Figure 5 plots the amount of wealth wasted against the level of risk aversion, for the cases of no constraints and only non-negativity constraints. As we can see, inefficiency is always lower in the second situation; in many cases, we observe that the benchmark

[10] An unconstrained BJS test, however, would not reject the null of efficiency for the benchmark, obtaining a statistic equal to -0.1037 with an associated p-value of 0.9174. The benchmark would actually provide a risk (4.13%) only slightly higher than the one (4.05%) of the optimal portfolio with the same expected return (0.73%).

wastes less than 0.5% of wealth. Note that the dashed lines represent the confidence intervals for the wealth wasted; the interval is smaller with constraints.

Figure 5.

Wealth wasted by the benchmark for different levels of relative risk aversion (%)



In table 4 we show the result of an efficiency test on the benchmark, where the null hypothesis is

$$H_0 : \lambda\left(\eta_b, \sigma_b^2, \eta, \Sigma, \gamma\right) = 0$$

The wealth loss does not seem to change a great deal when adding further constraints; it is, instead, much more sensitive to the risk aversion parameter.

We see in table 4 that a t test of efficiency always rejects the null hypothesis. Below we show the simulated rejection rates taken from a Monte Carlo simulation of the primitive assets; details on Monte Carlo simulation are provided in §7. We find very small differences between p-values and simulated rejection rates with our test, no matter whether the results come from a closed-form or a numerical solution.

Table 4.

Test statistics and hypothesis testing - benchmark

Panel A: No constraints and Non-negativity constraints

	No Constraints					Non-Negativity Constraints				
%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
Wealth Loss	0.8984	0.6228	0.5289	0.6173	0.8855	0.4548	0.4489	0.4610	0.5593	0.8074
Std. error	0.3995	0.1987	0.1054	0.1027	0.1326	0.0782	0.0791	0.0697	0.0821	0.1137
Lower Conf. Int.	0.1123	0.2326	0.3222	0.4158	0.6254	0.3015	0.2937	0.3243	0.3982	0.5843
Upper Conf. Int.	1.6784	1.0114	0.7352	0.8185	1.1450	0.6079	0.6039	0.5976	0.7201	1.0301
Test*	2.2386	3.1248	5.0065	5.9905	6.6505	5.8056	5.6601	6.5968	6.7904	7.0718
P-Value	0.0252	0.0018	0	0	0	0	0	0	0	0
Rej. Rate	*0*	*0*	*0.0020*	*0*	*0*	*0*	*0*	*0*	*0*	*0*

Panel B: Equality constraints and Non-negativity plus Equality constraints

%	Equality Constraints					Non-Negativity and Equality Constraints				
	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
Wealth Loss	0.8905	0.6225	0.5226	0.5911	0.8160	0.4456	0.4409	0.4493	0.5377	0.7653
Std. error	0.3974	0.1987	0.1024	0.0960	0.1226	0.0726	0.0737	0.0671	0.0776	0.1087
Lower Conf. Int.	0.1086	0.2322	0.3217	0.4028	0.5753	0.3032	0.2963	0.3176	0.3854	0.5521
Upper Conf. Int.	1.6662	1.0113	0.7231	0.7790	1.0561	0.5877	0.5853	0.5808	0.6898	0.9781
Test*	2.2309	3.1225	5.0899	6.1395	6.6264	6.1259	5.9677	6.6770	6.9069	7.0148
P-Value	0.0257	0.0018	0	0	0	0	0	0	0	0
Rej. Rate	*0*	*0*	*0.0020*	*0*	*0*	*0*	*0*	*0*	*0*	*0*

* *Null hypothesis: wealth loss equal to zero.*

We can also derive the optimal coefficient of relative risk aversion, i.e., the coefficient that makes the performance of the benchmark as optimal as possible. We see in table 5 that the optimal γ is equal to a reasonable 4.5227[11]. To understand γ, consider the following experiment. An investor is given a choice of a fixed sum of money in the next period or a lottery that pays \$800 with a probability of 0.5 and \$1,200 with a probability of 0.5. A risk neutral investor would be indifferent between the actuarial value of the lottery, \$1,000, and the lottery. An investor with $\gamma = 3$ is indifferent between \$940 and the lottery, and an investor with $\gamma = 5$ is indifferent between \$900 and the lottery. Gollier (2002), furthermore, observes that γ levels higher than 10 are implausible. The 95 percent confidence interval becomes acceptable too. Using this coefficient, there is a wealth loss of 0.53%, and it is significantly different from zero; a statistical test, indeed, rejects the null hypothesis of $cv_0 = 0$, with both theoretical and empirical distributions. This means that there is no risk aversion coefficient for which the benchmark is at least as efficient as the optimal portfolio.

In general, we can conclude that the benchmark is inefficient, but this inefficiency turns out to be unexpectedly small, even if the benchmark is dominated by one of the primitive assets.

Table 5.

Optimal RRA coefficient - benchmark

Benchmark No constraints	Optimal RRA	S.E.	Lower Conf. Int.	Upper Conf. Int.	P-value	Rejection Rate
RRA	4.5227	1.1153	2.3366	6.7087	-	-
Wealth loss (%)	0.5275	0.1093	0.3130	0.7416	-	-
Test	4.8129	-	-	-	0.0000	0.0000

[11] It would be equal to 4.8050 with the equality constraint, 2.7949 with short-sale constraints and 2.9694 with short-sale and equality constraints. The last two values can be obtained only numerically; in both cases, the procedure ended after 7 iterations. In neither of these cases we are able to associate a standard error.

6.2. Portfolio case

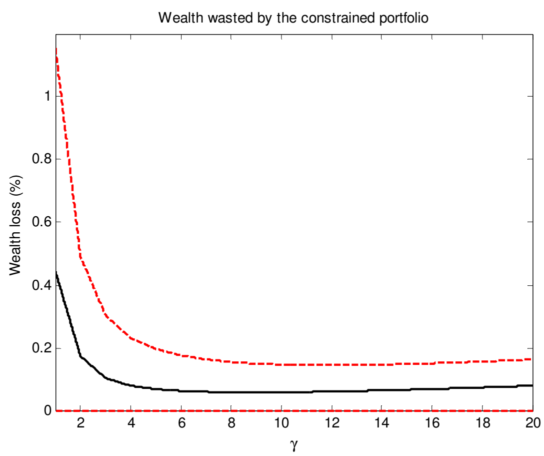
In the following section we consider an application of the portfolio version of our statistic. We analyze two cases; we first compare the unconstrained and the constrained optimal portfolios, to measure the cost of an additional constraint. We then consider equally-weighted portfolios, to establish how costly naïve strategies are.

COST OF ADDITIONAL CONSTRAINTS

In figure 6 we show the pattern of the wealth loss when comparing the optimal portfolio subject to short-sale constraints with the unconstrained optimal portfolio. The level of inefficiency decreases sharply after $\gamma = 2$, stabilizing soon below 0.1 percent. The lower confidence interval, however, is always equal to zero. It means that there is no evidence that adding non-negativity constraints worsens the efficiency.

Figure 6.

Wealth wasted by the constrained (non-negativity) portfolio for different levels of relative risk aversion



In table 6 we show the amount of wealth wasted when using a constrained optimal portfolio instead of the unconstrained optimal one. The wealth loss ranges from 0.06% to 0.45% with non-negativity constraints, from 0.0003% to 0.08% with equality constraints, and from 0.08% to 0.45% with both constraints. The wealth loss is smaller with only equality constraints, meaning that such restriction is able to explain a smaller part of the overall inefficiency. In no case, however, the lower bound of the confidence interval is higher that zero: in other words, we can never reject the null that the constrained optimal portfolio is inefficient, compared with the unconstrained optimal portfolio.

This approach also gives an idea of the cost of imposing additional constraints to a portfolio. It allows us to assess if the wealth loss in the presence of both constraints is significantly different from the wealth loss with only non-negativity constraints. We test indeed if an optimal portfolio with non-negativity and equality constraints is significantly less efficient than an optimal portfolio with only non-negativity constraints.

Table 6.

Test statistics and hypothesis testing – portfolio

Panel A: Non-negativity and Equality constraints

	NON-NEGATIVITY CONSTRAINTS					EQUALITY CONSTRAINTS (OTHER=10%)				
%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
WEALTH LOSS	0.4456	0.1746	0.0682	0.0584	0.0787	0.0080	0.0003	0.0064	0.0264	0.0701
STD. ERROR	0.3637	0.1615	0.0653	0.0449	0.0429	0.0486	0.0061	0.0197	0.0299	0.0404
LOWER CONF. INT.	0	0	0	0	0	0	0	0	0	0
UPPER CONF. INT.	1.1559	0.4906	0.1962	0.1464	0.1627	0.1033	0.0123	0.0451	0.0849	0.1493

Note: wealth loss is computed by comparing the optimal unconstrained portfolio with the optimal constrained portfolio.

Panel B: Non-negativity plus Equality constraints

	NON-NEGATIVITY AND EQUALITY CONSTRAINTS (OTHER=10%)				
%	γ=1	γ=2	γ=5	γ=10	γ=20
WEALTH LOSS	0.4549	0.1826	0.0800	0.0800	0.1211
STD. ERROR	0.3674	0.1651	0.0708	0.0527	0.0532
LOWER CONF. INT.	0	0	0	0	0.0168
UPPER CONF. INT.	1.1723	0.5057	0.2187	0.1832	0.2254
TEST*	0.0253	0.0485	0.1662	0.4108	0.7968
P-VALUE	0.9798	0.9613	0.8680	0.6812	0.4256
REJ. RATE	*0.8620*	*0.8260*	*0.7800*	*0.6940*	*0.3520*

* *Null hypothesis: wealth loss equal to the one with non-negativity constraints*

Looking at the last three rows in panel B of table 6, we can conclude that in no case there is a significant worsening in performance when adding the equality constraint to the non-negativity constraints. Imposing this additional constraint, then, is not relevant according to our test, even when its Monte Carlo distribution is used. This result is the opposite of what we would have been led to believe by looking at the optimal portfolios in table 2. The optimal portfolios with non-negativity and both constraints, do indeed, seem rather different, if only because the equality constraint is binding. This approach can in principle be applied when comparing any pair of nested portfolios[12]. We also measured the importance of any inequality constraint, and found that nearly all the variation in performance is the result of imposing two non-negativity constraints on just two assets: "manufacturing" and "other" markets.

[12] i.e., in which one portfolio is optimal under more restrictions than the other.

NAÏVE STRATEGY

Let us suppose now that the agent follows a naïve investment strategy, i.e., invests exactly the same amount of wealth in each of the ten assets. Such a portfolio is inefficient under a mean-variance analysis; a JK test run using the 10 industry portfolios, indeed, is worth 17.5876 with a p-value of 0.0403. We wonder, therefore, if this portfolio is still inefficient under the framework in this paper.

There are several reasons for studying a naïve portfolio. First, it is easy to implement because it does not require any estimation or optimization. Second, it is empirically proven (Benartzi and Thaler, 2001) how investors often continue to use such simple rules for allocating their wealth across assets. The literature deals with this portfolio, then, since it is simple to use and reasonably easy to implement assuming difficulty in diversifying (DeMiguel et al., 2005). In order to empirically arrange the portfolio composition as suggested by the theoretical models we need to know, indeed, the parameters of the model for a particular set of asset returns and then to solve for the optimal portfolio weights.

Figure 7 shows that the point estimate of wealth loss is below 0.4% for most levels of risk aversion, except for the smallest, where the wealth loss is higher, reaching a fairly large 1.3% when $\gamma = 1$. This suggests us that less risk averse individuals could take higher profit by investing in a portfolio different from the naïve. The standard error of the estimated wealth loss, nevertheless, is too large to support that conclusion. Observe, furthermore, how the result changes completely when we look at the wealth loss in the presence of inequality constraints: in this case, indeed, the point estimate is smaller for less risk averse individuals. In both cases, however, the lower bound of the confidence interval reaches the zero point for any $\gamma \leq 8$: for reasonable levels of risk aversion, therefore, we cannot reject the null of efficiency of a naïve portfolio.

Figure 7.

Wealth wasted by with the naïve portfolio for different levels of relative risk aversion

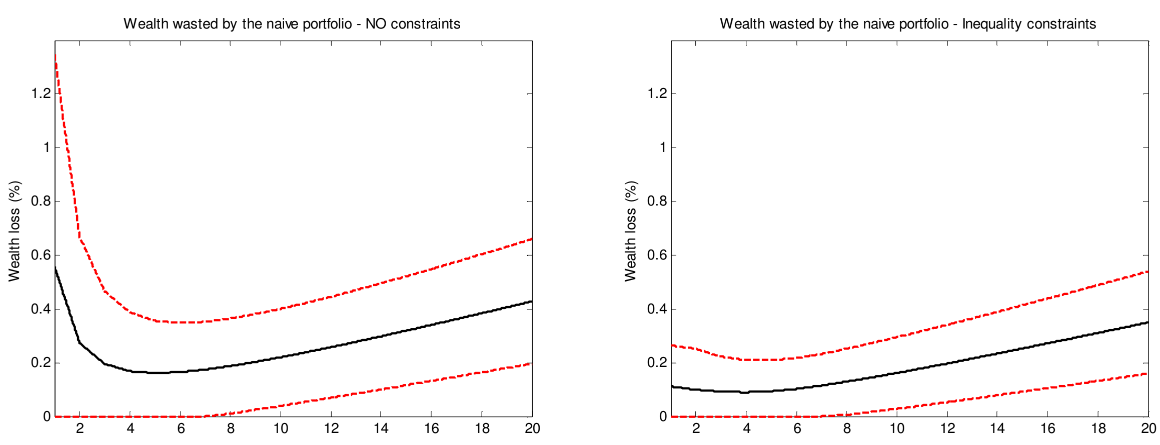


Table 7 provides us with the wealth wasted when holding this portfolio instead of the optimal under several constraints. The wealth however wasted is smaller (below 0.2%) for γ between 5 and 10; it grows for more elevate risk aversions and is much higher for smaller risk aversions, at least in the absence of any constraint. Observe, indeed, that when we consider in the analysis inequality constraints even the case of $\gamma = 1$ happens to get a very small amount of wealth loss. Given the lower boundary of the confidence interval, however, when $\gamma = 1$, 2 or 5 in no case we have enough evidence to conclude that the naïve strategy is not efficient.

We also report the results of the t test in which the null hypothesis assumes that the amount of wealth loss is equal to that obtained with no restrictions; using the theoretical distribution, just in two cases we have enough evidence for concluding that imposing more restrictions the naïve strategy becomes less inefficient: when we add inequality constraints, or both constraints, and $\gamma = 1$ or 2. This implies that, for reasonable levels of risk aversion, we cannot conclude that a naïve strategy is inefficient, and that, as we add more constraints, the point estimate of its wealth loss decreases significantly for low risk-averse individuals. In this case, therefore, accounting for market frictions helps explain much of the rationale behind the recourse to this strategy.

Table 7.

Test statistics and hypothesis testing – equally weighted naïve portfolio

Panel A: No constraints and Non-negativity constraints

	NO CONSTRAINTS					NON-NEGATIVITY CONSTRAINTS				
%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
WEALTH f	0.5568	0.2741	0.1616	0.2198	0.4280	0.1117	0.0997	0.0935	0.1615	0.3496
STD. ERROR	0.4050	0.2006	0.0996	0.0921	0.1182	0.0779	0.0777	0.0590	0.0680	0.0972
LOWER CONF. INT.	0	0	0	0.0391	0.1960	0	0	0	0.0281	0.1589
UPPER CONF. INT.	1.3475	0.6666	0.3569	0.4001	0.6595	0.2643	0.2519	0.2091	0.2947	0.5399
TEST*	-	-	-	-	-	-5.7259	-2.2471	-1.1545	-0.8575	-0.8076
P-VALUE	-	-	-	-	-	0	0.0246	0.2483	0.3911	0.4193
REJ. RATE	-	-	-	-	-	*0*	*0*	*0.2040*	*0.3480*	*0.3240*

Panel B: Equality constraints and Non-negativity plus Equality constraints

	EQUALITY CONSTRAINTS					NON-NEGATIVITY AND EQUALITY CONSTRAINTS				
%	γ=1	γ=2	γ=5	γ=10	γ=20	γ=1	γ=2	γ=5	γ=10	γ=20
WEALTH f	0.5489	0.2739	0.1552	0.1934	0.3581	0.1024	0.0917	0.0817	0.1399	0.3073
STD. ERROR	0.4025	0.2006	0.0970	0.0857	0.1090	0.0715	0.0714	0.0561	0.0628	0.0914
LOWER CONF. INT.	0	0	0	0.0253	0.1443	0	0	0	0.0167	0.1280
UPPER CONF. INT.	1.3346	0.6663	0.3451	0.3612	0.5715	0.2525	0.2316	0.1916	0.2629	0.4862
TEST*	-0.0198	-0.0013	-0.0657	-0.3077	-0.6416	-6.3664	-2.5571	-1.4242	-1.2727	-1.3226
P-VALUE	0.9842	0.9990	0.9476	0.7583	0.5213	0	0.0106	0.1544	0.2031	0.1860
REJ. RATE	*0.8740*	*0.8580*	*0.9520*	*0.8440*	*0.4740*	*0*	*0*	*0.0860*	*0.1340*	*0.0860*

* *Null hypothesis: wealth loss equal to the one in case of no restrictions.*

The overall impression, therefore, is that a naïve strategy is not a bad investment at all. This result is not surprising: Brennan and Torous (1999), for instance, conclude from a simulation analysis that even an equally weighted portfolio of as few as five randomly chosen firms can provide the same level of expected utility as the market portfolio. In DeMiguel et al. (2005), furthermore, this strategy performs quite well too. The authors suggest that, even if naïve diversification results in a lower performance than optimal diversification, the loss is smaller than the one arising from having to use as inputs for the optimizing models parameters that are estimated with error rather than known precisely. In our analysis we do not have enough information to reject the null of efficiency. We are led to believe, furthermore, that even the point estimate of the wealth loss is so small that it is actually cheaper than any cost of information search and, then, many investors would prefer this solution to a theoretically more efficient portfolio.

Lastly, we see in table 8 that the optimal γ derived using equation (10) takes a value of 5.0680[13] and however not higher than 7.3688 in a 95 percent confidence interval. In other words, the agent who gets the smaller wealth loss has a risk aversion roughly equal to about $\gamma = 5$. The corresponding wealth loss is 0.1616 percent, and its confidence interval produces a lower bound just equal to zero. The wealth loss is, however, significantly different from zero. In this case, as before, the optimal risk aversion is able to reduce the inefficiency, but not sufficiently so to justify the efficiency of the actual investment.

Table 8.

Optimal RRA coefficient – equally weighted naïve portfolio

Portfolio No constraints	Optimal RRA	S.E.	Lower Conf. Int.	Upper Conf. Int.
RRA	5.0680	1.1739	2.7672	7.3688
Wealth loss (%)	0.1616	0.0990	0	0.3554

7. Empirical distribution of the test

In this section we study how our statistic performs in small samples. The statistic $\ell_b = \ell\left(e_b, s_b^2, e, S, \gamma\right)$ is a highly non-linear function of the random variables and the small sample distribution of the test can be significantly different from its normal asymptotic distribution. The knowledge of the test small sample properties can have relevant implication for the empirical analysis. For instance, in Bucciol (2003) the author makes use of a statistic closely related to the ones used in Basak et al. (2002) and adopted in this paper; partly because of a small sample size, he obtains a

[13] 3.8976 with short-sale constraints, 5.5095 with the equality constraint and 4.1191 with short-sale and equality constraints.

generalized efficiency of Italian household portfolios, apparently too wide to be explained only by means of inequality constraints.

To establish the small-sample properties of our test we perform a Monte Carlo simulation. Given the time series for returns on primitive assets, with sample moments e and S, and the time series for returns on benchmark, with sample moments e_b and s_b^2, we adopt the following algorithm:

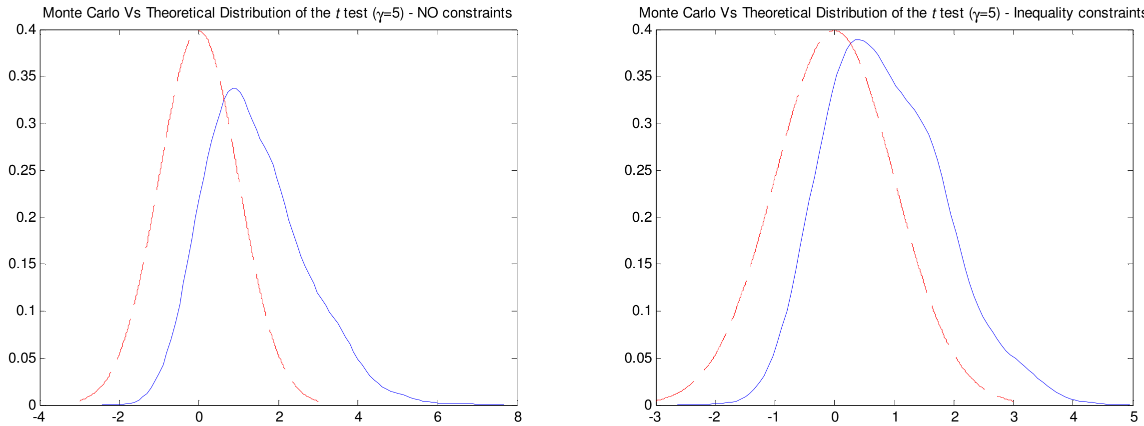
1. Determine the wealth loss $cv_0 = cv_0\left(e_b, s_b^2, e, S, \gamma\right)$ wasted or generated by the benchmark relative to the optimal portfolio; therefore $\ell_0 = -\log\left(1 - cv_0\right)$ is the numerator of the test statistic. Also derive its variance $V_0 = \frac{1}{T}\hat{V}\left(e_b, s_b^2, e, S, \gamma\right)$;
2. Repeat the following a number N of times:
   - 2.1. Generate *new* time series of length T from a multivariate normal distribution whose parameters are the moments e, S, e_b, s_b^2 (both primitive assets and benchmark);
   - 2.2. The new time series lead to new moments $e^*, S^*, e_b^*, s_b^{2*}$ and, as a consequence, to new values for the optimal portfolio, the wealth loss $cv_i^*\left(e_b^*, s_b^{2*}, e^*, S^*, \gamma\right)$ and the function $\ell_i^* = -\log\left(1 - cv_i^*\right)$.
   - 2.3. Compute the test statistic $t_i^* = \frac{\ell_i^* - \lambda_0}{\left(V_0\right)^{1/2}}$

In this paper we show results with $N = 1000$ and, if not otherwise specified, $T = 664$; smaller or larger values of N do not seem to provide significant differences. In order to avoid possible errors due to undetected autocorrelation, we assume absence of autocorrelation[14]. Below we show the results for portfolios relative to $\gamma = 5$, with no constraint or only non-negativity constraints. Simulations for other cases do not provide significantly different results. Figure 8 shows the empirical and the theoretical distributions for the benchmark test.

[14] The autocorrelation would enter the matrix $\hat{V}$ through the HAC covariance estimate.

Figure 8.

Empirical (solid line) Vs. theoretical (dashed line) distribution for the benchmark test



Using this procedure we realize that 1) the empirical statistic actually appears to be normally distributed, 2) the estimated variance correctly replicates the true variance, especially in the constrained case, but 3) the empirical distribution is not centered around zero. The test average is actually higher than zero, with an average value which decreases as γ increases (see table 9 for the benchmark test). Analogous results come from the analysis of the distribution of the wealth loss.

Table 9.

Average value for the benchmark test under the null hypothesis Monte Carlo simulation, $T = 664$.

Constraints	γ=1		γ=2		γ=5		γ=10		γ=20	
	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$
NO	0.6954	1.7250	0.3500	1.7507	0.1446	1.3659	0.0792	0.7658	0.0522	0.3903
INEQUALITY	0.0980	1.2478	0.0788	0.9918	0.0610	0.8708	0.0405	0.4903	0.0260	0.2272
EQUALITY	0.6228	1.5533	0.3121	1.5605	0.1274	1.2373	0.0686	0.7102	0.0444	0.3592
BOTH	0.0890	1.2215	0.0713	0.9672	0.0560	0.8306	0.0370	0.4737	0.0236	0.2150

Note: $E[\ell]-\lambda$ in percentage scale.

This effect, however, tends to disappear as $T \to \infty$. For instance, by setting $T = 2000$, we have a much smaller distortion:

Table 10.

Average value for the benchmark test under the null hypothesis Monte Carlo simulation, $T = 2000$.

Constraints	γ=1		γ=2		γ=5		γ=10		γ=20	
	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$
NO	0.2321	0.5756	0.1162	0.5812	0.0471	0.4449	0.0249	0.2406	0.0152	0.1136
INEQUALITY	0.0355	0.4516	0.0252	0.3171	0.0208	0.2971	0.0123	0.1493	0.0067	0.0585
EQUALITY	0.2069	0.5160	0.1035	0.5177	0.0420	0.4079	0.0222	0.2293	0.0136	0.1103
BOTH	0.0331	0.4547	0.0228	0.3077	0.0193	0.2855	0.0114	0.1459	0.0064	0.0584

Note: $E[\ell]-\lambda$ in percentage scale.

This phenomenon depends on the application of the delta method to the highly non-linear function $\ell_b = \ell\left(e_b, s_b^2, e, S, \gamma\right)$. The delta method, indeed, makes use of a first-order Taylor expansion, from which

(12) $$\ell_b = f\left(\overline{X}_T, \gamma\right) \cong f\left(X, \gamma\right) + \nabla\left(\gamma\right)'\left(\overline{X}_T - X\right)$$

and $\nabla\left(\gamma\right)$ is the gradient of $f\left(\overline{X}_T, \gamma\right)$, following the notation in §3. From equation (12) the expectation is worth

$$E\left[\ell_b\right] \cong f\left(X, \gamma\right) + E\left[\nabla\left(\gamma\right)'\left(\overline{X}_T - X\right)\right] = f\left(X, \gamma\right) = \lambda$$

being $\overline{X}_T$ an unbiased estimator of X; applying the delta method we assume, therefore, that the statistic is unbiased. The Monte Carlo simulation, however, provides evidence that this simplification works poorly in small samples: if we considered a second-order Taylor expansion, indeed, we would get that

$$\ell_b = f\left(\overline{X}_T, \gamma\right) \cong f\left(X, \gamma\right) + \nabla\left(\gamma\right)'\left(\overline{X}_T - X\right) + \frac{1}{2}\left(\overline{X}_T - X\right)'\nabla^2\left(\gamma\right)\left(\overline{X}_T - X\right)$$

with $\nabla^2\left(\gamma\right)$ hessian of $f\left(\overline{X}_T, \gamma\right)$. Taking its expectation,

(13) $$E\left[\ell_b\right] \cong f\left(X, \gamma\right) + \frac{1}{2}E\left[\left(\overline{X}_T - X\right)'\nabla^2\left(\gamma\right)\left(\overline{X}_T - X\right)\right]$$

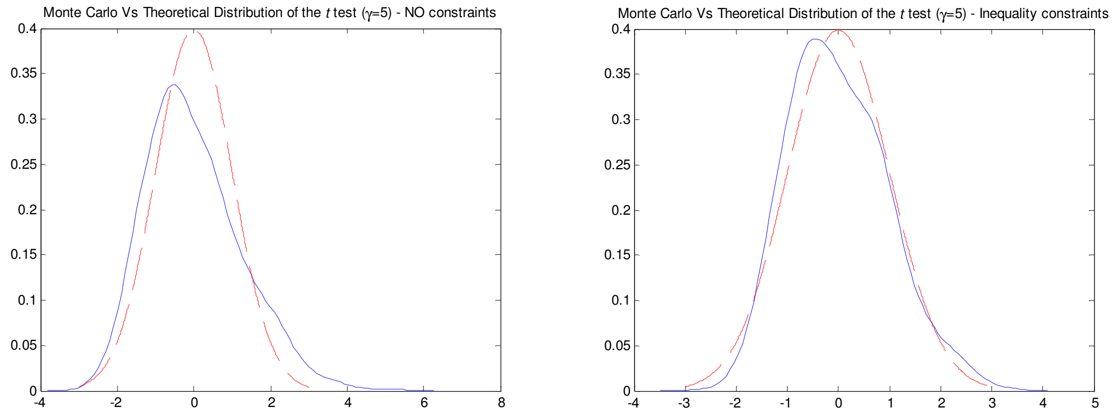
where the second-order term does not disappear. We do not go further and make $\nabla^2\left(\gamma\right)$ explicit since it is the derivative of $\nabla\left(\gamma\right)$, a complicated function of X and, above all, the optimal weights w^*, whose relation with X is in most cases unknown.

Once we correct for the Monte Carlo sample average of the test, nevertheless, we are able to almost perfectly replicate the asymptotic distribution[15]. Using this simulated distribution, once corrected for the small sample bias, we compute the rejection rates of the tests in §6 and observe how they are close to the theoretical p-values.

[15] A formal Kolmogorov-Smirnov test does not reject the null of efficiency in any case but when constraints are absent for small levels of risk aversion.

Figure 9.

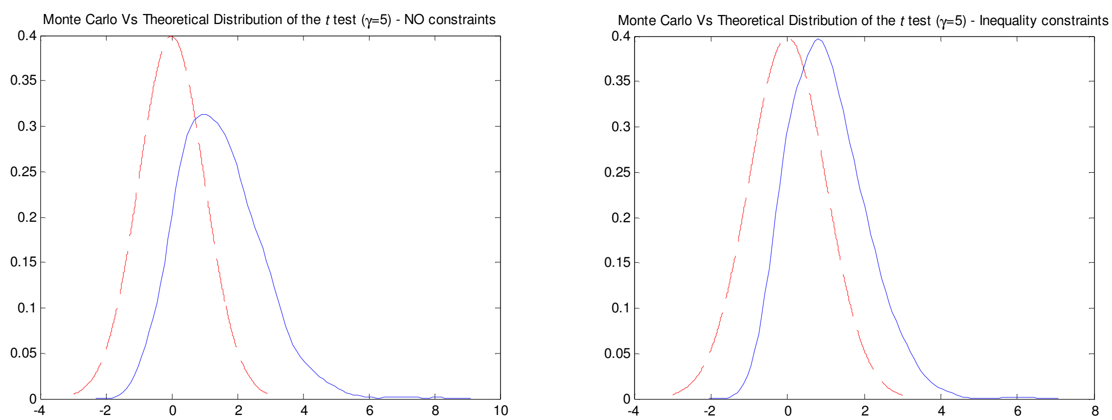
Empirical (solid line) Vs. theoretical (dashed line) centered distribution for the benchmark test



For the portfolio test, using an analogous simulation procedure with the naïve portfolio, we get similar results; as before, we observe that the approximation works better in the presence of inequality constraints, where otherwise there seems to be an underestimate of the asymptotic variance.

Figure 10.

Empirical (solid line) Vs. theoretical (dashed line) distribution for the portfolio test



The numerical assessment of the bias is reported on table 11; the bias is the same as in equation (13).

Table 11.

Average value for the portfolio test under the null hypothesis

Monte Carlo simulation

Constraints	γ=1		γ=2		γ=5		γ=10		γ=20	
	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$	$E[\ell]-\lambda$	$E[t]$
NO	0.7122	1.7486	0.3597	1.7878	0.1499	1.5030	0.0828	0.8973	0.0547	0.4607
INEQUALITY	0.0958	1.2277	0.0771	0.9907	0.0604	1.0230	0.0411	0.6040	0.0276	0.2826
EQUALITY	0.6327	1.5632	0.3200	1.5908	0.1339	1.3785	0.0744	0.8667	0.0494	0.4515
BOTH	0.0872	1.2180	0.0691	0.9742	0.0557	0.9910	0.0378	0.6014	0.0253	0.2761

Note: $E[\ell]-\lambda$ in percentage scale.

It seems, therefore, preferable to use the empirical distribution on data analyses or, in alternative, the theoretical distribution corrected for the bias in small samples.

Further simulation studies, conducted using the same benchmark against a set of as many as 30 or just 5 primitive assets, industry portfolios taken from the same source, show that the empirical distribution of this test keeps similar, These simulation results are therefore robust to different number of primitive assets.

8. Conclusion

In this paper we study the efficiency of a benchmark or a portfolio in an expected utility framework, dealing with complex problems in which the optimal portfolio depends on weight constraints. We consider a measure of compensative variation which reads as the wealth loss between optimal and sub-optimal portfolios. We provide its asymptotic distribution and discuss the related inefficiency test. We suggest an estimation strategy for the risk aversion parameter based on the parameter value that minimizes the wealth loss with respect to the optimal portfolio. This estimate could turn out to be interesting when establishing, for instance, the implicit risk aversion adopted by fund managers when building their fund portfolio. The statistic can flexibly deal with equality and inequality constraints on portfolio composition, even if the presence of inequality constraints makes it impossible to derive a closed-form solution.

Although we depart from the classical literature of mean-variance analysis, we show that the two frameworks are comparable and to some extent provide analogous results; in particular, the optimal portfolios without inequality constraints differ only for a normalizing factor.

We find the asymptotic distribution for the test and discuss its small sample properties: given the results of our Monte Carlo simulations, we believe that a better way to make use of this statistic is to consider its empirical rejection rates, drawn from a Monte Carlo simulation, instead of theoretical p-values.

Our empirical application, based on ten industry portfolios for the U.S. market, shows that there is no enough evidence to reject the null of efficiency for a naïve investment strategy with reasonable values of the risk aversion coefficient. The point estimates of the wealth loss, furthermore, are rather small, often about 0.10%, and it seems that considering inequality constraints into the analysis really helps explain such an apparently inefficient behavior when the risk aversion parameter is low. Our conclusion confirms the results in Brennan and Torous (1999) and Das and Uppal (2004). When using a benchmark, such as the S&P 500, that in the relevant period is dominated by at least one industry portfolio, our test concludes for its inefficiency, but this inefficiency is unexpectedly small. In all cases the wealth loss is not higher than 0.5 point percentage, and the optimal level of risk aversion is a reasonable 5.

In our agenda, we plan to correct the distribution of our statistic for the small sample bias using a block bootstrap technique, and to focus on the size of the implicit risk aversion parameter used by fund managers when choosing the composition of the fund portfolio. Finally, a promising further step is to consider a long term perspective and then analyze within this framework the behavior of forward-looking agents with regards to their lifetime portfolios.

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