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HOPF BIFURCATION FOR THE DISCRETE – DELAY KALDOR-KALECKI MODEL

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Hopf bifurcation for the discrete – delay Kaldor - Kalecki model

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The present work will focus on a Kaldor - Kalecki non – linear business cycle model in income and capital, with discrete time and delay argument characteristics. What it will state, considering an investment function similar to the one proposed by Rodano and using the linear approximation analysis, are the local stability property and local bifurcations conditions, given the parameter space. Numerical examples will be given in the end, to support the theoretical results obtained.

Key Words: business cycle, Hopf bifurcation, discrete-delay time.

1. INTRODUCTION

There were many alternative formulations throughout the economic literature and not only, regarding the model of the business cycle. The first to proposed a formulation of this model was Kaldor, who considered the basic nonlinear specification for the business cycle model. Nonetheless, considering the case of actual dynamic economics, this framework can no longer be regarded as the benchmark.

The benchmark approach of Kaldor was consequently developed by Chang and Smith who introduced the concept of continuous time as articulated by a system of two nonlinear differential equations in income and capital, creating the dynamic system framework of the Kaldor model. The normal evolution at this stage of the model was analyzed by Dana and Malgrange in 1984, Hermann in 1985, Lorenz in 1992 and 1993 and was given by the development of the discrete-time dynamic system as conveyed by two nonlinear differential equations system.

To summarize, the essential idea of the business cycle model of Kaldor is that if propensity to invest is superior to propensity to save, the system is unstable and it produces a set of fluctuations. On the other hand, if the system is remote from the equilibrium point, propensity to invest decreases until it becomes inferior to the propensity to save. Consequently, when analyzing this type of model, we have to regard not only the speed of reaction to excess demand, which has a destabilizing effect, but also the propensity to save, knowing that it has a stabilizing role.

The next development of the model was proposed by Bischi et all. [1] who considered the case of a discrete dynamic system form, by assuming that the factor determining firms' investment decisions is the expected "normal" value of income, exogenously given. Bischi et all. [1] analyzed the joint dynamic effects of the speed of reaction to excess demand and propensity to save and showed that "the exogenously given equilibrium is only stable for low values of the firms' speed of reaction and sufficiently high values of the propensity to save". Furthermore, for sufficiently high values of the speed of adjustment, the dynamic scenario strongly depended on the values of the propensity to save with a low level of propensity to save generating bi-stability.

The work concentrates on a Kaldor-Kalecki model, that models the $(n+1)$-moment income and capital, starting from considering the n-moment income and capital and $(n-m)$-moment income, where $m \geq 0$. If $m = 0$, the model represents the discrete-time Kaldor model, as described and analyzed by Bischi [1]. If $m = 1$, the model will be analyzed in the present work. The paper proceeds as follows: Section 2 introduces the discrete-delay Kaldor model. In section 3 we analyze the roots of characteristic equation for the $m = 0$ and $m = 1$ cases, function of the adjustment parameter s. Using variable transformation, we establish the existence of a value s_0 which is Hopf bifurcation. In section 4 and 5 we describe the normal form for $m = 0$ and $m = 1$, as well as the orbits of state variables. By using the software Maple11, keeping the values of p, q, r fixed, we will offer practical support for the theoretical results obtained so far. Section 6 concludes with respect to the obtained results' importance and presents future possible research analysis for this model's case. Appendix A will describe the discrete - delay Kaldor model, regarding the investment function presented by Rodano and the saving function considered by Keynes. We establish the model's Jacobian matrix in a fix point, the characteristic equation and the eigenvectors $q \in \mathbb{R}^{m+2}$, $p \in \mathbb{R}^{m+2}$associated to the eigenvalues.

2. THE DISCRETE-DELAY KALDOR MODEL

This section presents the discrete - delay Kaldor model of business cycle for the state variables of income (national income) Y_n and capital stock K_n, where $n \in \mathbb{N}$, as described and analyzed by Dobrescu et all. [2]. This model corresponds to a system of equations with discrete time and delay, given by:

$$\begin{aligned} Y_{n+1} &= Y_n + s\,[I(Y_n, K_n) - S(Y_n, K_n)] \\ K_{n+1} &= K_n + I(Y_{n-m}, K_n) - qK_n \end{aligned} \tag{1}$$

Where: Y_{n-m} describes income on moment $n-m$, with $m \geq 1$, $I : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ represents the investment function and $S : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ represents the savings function, both differentiable functions; s represents an adjustment parameter, which describes the reaction of the system to the difference between investment and saving.[1] Assuming Keynes's hypothesis which states that the saving function is proportional with income, the proportionality parameter being $p \in (0, 1)$, the saving propensity, and also that investment function I is defined à la Rodano with respect to a certain normal level of income u, and a normal level of capital stock - $\frac{pu}{q}$, where $u > 0$, $q \in (0, 1)$, the following statements take place:

[1] For a thorough description of the model, please see the details in Dobrescu, Opris (2007) or Appendix A.

PROPOSITION 1. *(i). The Jacobi matrix of the application in the fix point $(y_0, ...y_0, \mathbf{k}_0)^T$ is the following:*

$$A = \begin{pmatrix} 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & ... & 1 & 0 \\ 0 & ... & ... & 1 + a_{10} & a_{01} \\ b_{10} & ... & ... & 0 & 1 + b_{01} \end{pmatrix} \tag{2}$$

(ii). The characteristic equation of matrix A is:

$$\lambda^{m+2} - a(s)\lambda^{m+1} + b(s)\lambda^m - c(s) = 0 \tag{3}$$

where $a(s) = 2 + a_{10} + b_{01},\ b(s) = (1 + a_{10})(1 + b_{01}),\ c(s) = a_{01}b_{10}$

(iii). The eigenvector $q \in \mathbb{R}^{m+2}$, that corresponds to the eigenvalue μ of matrix A has the components:

$$q_i = \mu^{i-1}, i = 2, .., m + 1, q_{m+2} = \frac{b_{10}}{\mu - 1 - b_{01}} \tag{4}$$

The eigenvector $p \in \mathbb{R}^{m+2}$, that corresponds to the eigenvalue $\overline{\mu}$ of matrix A^T has the components:

$$p_1 = \frac{(\overline{\mu} - 1 - a_{10})(\overline{\mu} - 1 - b_{10})}{m(\overline{\mu} - 1 - a_{10})(\overline{\mu} - 1 - b_{01}) + \overline{\mu}(2\overline{\mu} - 2 - a_{10} - b_{01})} \tag{5}$$

$p_i = \frac{1}{\overline{\mu}^{i-1}}p_1,\ \ i = 2, .., m - 2,\ \ \ p_{m+1} = \frac{1}{\overline{\mu}^{m-1}(\overline{\mu}-1-a_{10})}p_1,\ \ \ p_{m+2} = \frac{\overline{\mu}}{b_{10}}p_1$

The vectors q, p given by (3) and (4) satisfy the relationship

$$\sum_{i=1}^{m+2} q_i\overline{p_i} = 1$$

where

$$\begin{aligned} \rho_1 &= f'(y_0 - u), \rho_2 = f''(y_0 - u), \rho_3 = f'''(y_0 - u) \\ a_{10} &= s(\rho_1 - p), a_{01} = -rs, b_{10} = \rho_1, b_{01} = -q - r \end{aligned} \tag{6}$$

3. THE ANALYSIS OF THE CHARACTERISTIC EQUATION IN THE FIX POINT

In this section, the roots of the characteristic equation (3) are analyzed, function of the adjustment parameter s, for $m = 0$ and $m = 1$. For $m \geq 2$, the analysis raises some difficulties to be performed.

PROPOSITION 2. *If $m = 0$, the following affirmations are true:*
(i). The characteristic equation (3) becomes :

$$\lambda^2 - a(s)\lambda + b(s) - c(s) = 0 \tag{7}$$

(ii). If $q+r<1, \rho_1<1+\frac{r(q+r-4)}{(q+r-2)^2}$ and $s=s_0$, where

$$s_0=\frac{q+r}{(1-q-r)(\rho_1-p)+r} \tag{8}$$

then equation (7) has the complex roots in absolute value equal with 1.

(iii). Taking into consideration the variable change:

$$s(\beta)=s_0+\frac{(1+\beta)^2-b_0+c_0}{(\rho_1-p)(1-q)+rp} \tag{9}$$

where:

$$a_0=a(s_0), b_0=b(s_0), c_0=c(s_0)$$

equation (7) can be write as:

$$\lambda^2-a(\beta)\lambda+(1+\beta)^2=0 \tag{10}$$

where

$$a(\beta)=a_0+(\rho_1-p)\frac{(1+\beta)^2-b_0+c_0}{(\rho_1-p)(1-q)+rp} \tag{11}$$

Equation (10) has the roots:

$$\mu_{12}(\beta)=(1+\beta)\,e^{\pm i\theta(\beta)}, \theta(\beta)=\arccos\frac{a(\beta)}{2\,(1+\beta)^3} \tag{12}$$

for $|\beta|$ small enough.

(iv). The eigenvector $q\in\mathbb{R}^2$ which corresponds to the eigenvalue $\mu=\mu(\beta)$, considering the matrix A is:

$$q_1=1, q_2=\frac{\mu-1-a_{10}}{a_{01}} \tag{13}$$

The eigenvector $p\in\mathbb{R}^2$ which corresponds to the eigenvalue $\overline{\mu}=\overline{\mu}(\beta)$, considering the matrix A^T is:

$$p_1=\frac{a_{01}b_{10}}{a_{01}b_{10}+(\overline{\mu}-1-a_{10})^2}, p_2=\frac{a_{01}(\overline{\mu}-1-a_{10})}{a_{01}b_{10}+(\overline{\mu}-1-a_{10})^2} \tag{14}$$

PROPOSITION 3. *If $m=1$, the following affirmations are true:*

(i). The characteristic equation (3) becomes:

$$\lambda^3-a(s)\lambda^2+b(s)\lambda-c(s)=0 \tag{15}$$

(ii). The necessary and sufficient condition for equation (15) to admit two complex roots with absolute value equal to 1 and one root with absolute value less than 1, is that it exists $s_0\in\mathbb{R}$ such that:

$$|c_0|<1, |a_0-c_0|<2, b_0=1+a_0c_0-c_0^2 \tag{16}$$

with:

$$a_0=a(s_0), b_0=b(s_0), c_0=c(s_0) \tag{17}$$

(iii). Considering:

$$\alpha_1=\rho_1-p, \beta_1=2-q-r, \alpha_2=(\rho_1-p)(1-q-r), \beta_2=1-q-r, \alpha_3=-r\rho_1$$

For $|\beta|$ small enough, if the following expression is satisfied:

$$\left((a_0\alpha_3 + c_0\alpha_1)(1+\beta)^2 - \alpha_2(1+\beta)^4 - 2c_0\alpha_3\right)^2 - 4\left(\alpha_3\alpha_1(1+\beta)^2 - \alpha_3^2\right) *$$

$$* \quad \left(a_0c_0(1+\beta)^2 - c_0^2 - b_0(1+\beta)^4 + (1+\beta)^6\right) \geq 0 \tag{18}$$

then there is a function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$, $g'(0) = 0$ such that the variable change:

$$s = s_0 + g(\beta) \tag{19}$$

transforms equation (15) into the following equation:

$$\lambda^3 - a(\beta)\lambda^2 + b(\beta)\lambda - c(\beta) = 0 \tag{20}$$

with

$$a(\beta) = a_0 + \alpha_1 g(\beta), b(\beta) = b_0 + \alpha_2 g(\beta), c(\beta) = c_0 + \alpha_3 g(\beta) \tag{21}$$

Function $g(\beta)$ is given by equation:

$$\left(\alpha_3\alpha_1(1+\beta)^2 - \alpha_3^2\right) g^2 + \left(a_0\alpha_3(1+\beta)^2 + c_0\alpha_1(1+\beta)^2 - 2c_0\alpha_3 - \alpha_2(1+\beta)^4\right) g +$$

$$+ \quad a_0c_0(1+\beta)^2 - c_0^2 - b_0(1+\beta)^4 + (1+\beta)^6 = 0 \tag{22}$$

and equation (20), has the roots :

$$\mu_{12}(\beta) = (1+\beta)\, e^{\pm i\theta(\beta)}, \lambda_1(\beta) = \frac{c(\beta)}{(1+\beta)^2} \tag{23}$$

with:

$$\theta(\beta) = \arccos \frac{a(\beta)(1+\beta)^2 - c(\beta)}{2(1+\beta)^3} \tag{24}$$

Based on Proposition 2 and Proposition 3, in order to study the roots of the characteristic equation, it is equivalent to examine the transformed equation in function of β. Equations (12) and (23) yield that $\beta = 0$ is the point of Hopf bifurcation, and together with equation (19), find that the point that satisfies $s = s_0$ is the system's Hopf bifurcation point.

4. THE NORMAL FORM OF SYSTEM (3) IF $M = 0$

If $m = 0$, Kaldor model for which we made the translation $Y \to y + y_0$ and $K \to k + k_0$ is:

$$\begin{aligned} Y_{n+1} &= -spy_n - rsk_n + sf(y_n + y_0 - u) - f(y_0 - u) \\ K_{n+1} &= -(r+q)k_n + f(y_n + y_0 - u) - f(y_0 - u) \end{aligned} \tag{25}$$

Proposition 4. *Using the method from Kuznetsov [4] and Ford et all. [3], the following affirmations are true:*
(i). The canonical form of system (25) is as follows:

$$z_{n+1} = \mu(\beta)z_n + \frac{1}{2}\left(s(\beta)p_1 + p_2\right)\rho_2\left(z_n^2 + 2z_n\overline{z_n} + \overline{z_n}^2\right) + \\ +\frac{1}{6}\left(s(\beta)p_1 + p_2\right)\rho_3(\beta)\left(z_n^3 + 3z_n^2\overline{z_n} + 3z_n\overline{z_n}^2 + \overline{z_n}^3\right) \quad (26)$$

where $\mu = \mu(\beta)$ is given by equation (12), $s(\beta)$ is given by equation (9), p_1, p_2 is given by equation (14) and $z_n \in \mathbb{C}^2$.
(ii). The coefficient $c_1(\beta)$ associated to form (26) is:

$$c_1(\beta) = \left(\frac{\left(s(\beta)p_1 + p_2\right)^2\left(\overline{\mu}(\beta) - 3 + 2\mu\right)}{2\left(\mu^2 - \mu\right)\left(\overline{\mu} - 1\right)} + \frac{\left(s(\beta)p_1 + p_2\right)^2}{1 - \overline{\mu}} + \frac{s(\beta)p_1 + p_2}{2\left(\mu^2 - \overline{\mu}\right)}\right)\rho_2^2 + \\ +\frac{s(\beta)p_1 + p_2}{2}\rho_3 \quad (27)$$

(iii). Let us consider $I_1(0) = \mathrm{Re}(c_1(0)e^{-i\theta(0)}$, where $\theta(0)$ is given by (12). If $I_1(0) < 0$ in the neighborhood of the fixed point (y_0, k_0) there is a closed stable curve. If $I_1(0) > 0$, the curve is closed but unstable.
(iv). The solution of Kaldor system, in the neighborhood of the fixed point is:

$$Y_n = y_0 + z_n + \overline{z_n}, K_n = k_0 + q_2 z_n + \overline{q_2 z_n} \quad (28)$$

where z_n is a solution of equation (18). The investment function and the savings function are given by:

$$I_n = I(Y_n, K_n), S_n = qY_n \quad (29)$$

Using the formulas from Proposition 4, through the software called Maple11, for fixed values of p, q, r we obtain the results showed in the graph below. If $q = 0.2, r = 0.4, p = 0.5$, s_0 given by (8) and $f(x) = tan(x)$,then the trajectory is represented in the phase plane (Y, K) and $I_1(0) = 0.36315$, as it can be noticed in the following map (see Fig. 1.0. (Y_n, K_n)).

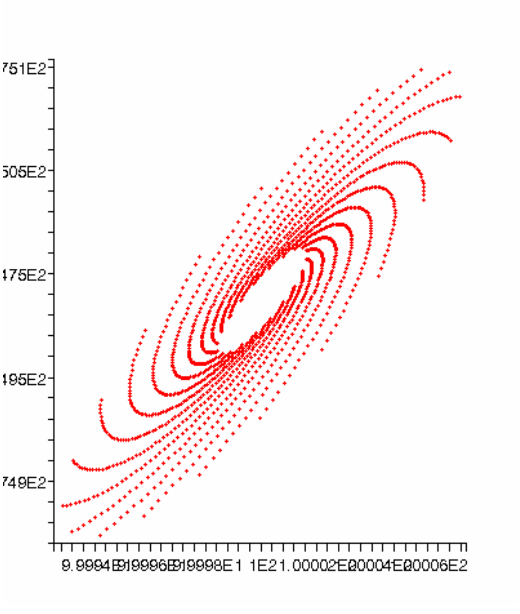
5. THE NORMAL FORM OF SYSTEM (3) IF $M = 1$

If $m = 1$, with the translation $Y \to y + y_0$ and $K \to k + k_0$, the Kaldor model becomes:

$$\begin{aligned} Y_{n+1} &= -spy_n - rsk_n + sf(y_n + y_0 - u) - f(y_0 - u) \\ K_{n+1} &= -(r+q)k_n + f(y_n + y_0 - u) - f(y_0 - u) \end{aligned} \quad (30)$$

Proposition 5. *The following affirmations are true:*
(i). The normal form associated to system (30) yields:

$$z_{n+1} = \mu(\beta)z_n + \frac{1}{2}\left(s(\beta)p_2 + p_3\right)\rho_2\left(z_n^2 + 2z_n\overline{z_n} + \overline{z_n}^2\right) + \frac{1}{2}g_{21}(\beta)z_n^2\overline{z_n} \quad (31)$$



with:

$$\begin{aligned}
g_{21}(\beta) &= \rho_2\left(s(\beta)p_2+p_3\right)\left(w_{20}\overline{q_2}+2q_2w_{11}\right)+\rho_3\left(s(\beta)p_2+p_3\right)\\
w_{20} &= \mu(\beta)^2\frac{s(\beta)\rho_2(1+q+r-rs(\beta))}{[\mu(\beta)^2-s(\beta)(\rho_1-p)]\left[\mu(\beta)^2+q+r\right]\mu(\beta)^2-\rho_1 rs(\beta)}\\
w_{11} &= \frac{-s(\beta)\rho_2(1+q+2r)}{[1-s(\beta)(\rho_1-p)]\left[1+q+r\right]\mu(\beta)^2+\rho_1 rs(\beta)}\\
q_2 &= \mu(\beta)\\
p_1 &= \frac{\left(\overline{\mu}(\beta)-1-s(\beta)\left(\rho_1-p\right)\right)\left(\overline{\mu}(\beta)-1-\rho_1\right)}{\begin{array}{c}[(\overline{\mu}(\beta)-1-s(\beta)\left(\rho_1-p\right))\left(\overline{\mu}(\beta)-1+q+r\right)+\\ +\overline{\mu}(\beta)\left(2\overline{\mu}(\beta)-2-s\left(\rho_1-p\right)+q+r\right)]\end{array}}\\
p_2 &= \frac{1}{\overline{\mu}(\beta)-1-s\left(\rho_1-p\right)}p_1\\
p_3 &= \frac{\overline{\mu}(\beta)}{\rho_1}p_1
\end{aligned} \tag{32}$$

and $s(\beta)$ is given by (19) and $\mu(\beta)$ is given by (23).

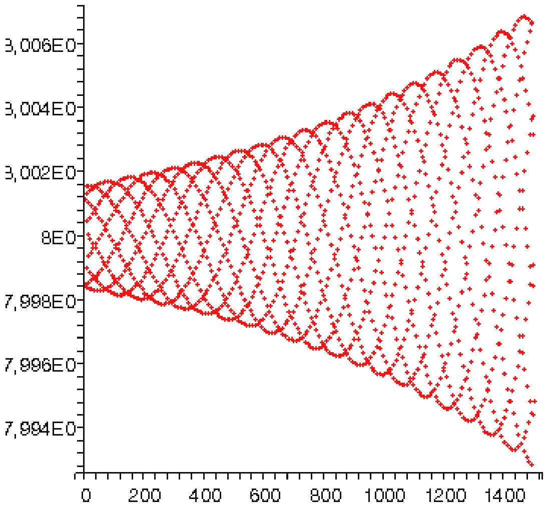
(ii). The solution of the system (30) in the neighborhood of the fixed point $(y_0, y_0, k_0) \in \mathbb{R}^3$ is :

$$\begin{aligned}
Y_n &= y_0+q_2z_n+\overline{q_2z_n}+k_1q_2^n\\
K_n &= k_0+q_3z_n+\overline{q_3z_n}+k_1q_3^n\\
U_n &= y_0+z_n+\overline{z_n}+k_1
\end{aligned} \tag{33}$$

with $k_1 \in \mathbb{R}$, with the investment function and the savings function :

$$I_n = I(Y_n, K_n), S_n = qY_n \tag{34}$$

Fig. 1.(n,Yn)



where $z_n \in \mathbb{C}^2$ is solution of equation (31).

(iii). The coefficient $c_1(\beta)$ associated to the equation (31) is:

$$c_1(\beta) = \left(\frac{(s(\beta)p_2 + p_3)^2 (\overline{\mu}(\beta) - 3 + 2\mu(\beta))}{2(\mu(\beta)^2 - \mu(\beta))(\overline{\mu}(\beta) - 1)} + \frac{|s(\beta)p_2 + p_3|^2}{1 - \overline{\mu}(\beta)} + \frac{|s(\beta)p_2 + p_3|}{2(\mu(\beta)^2 - \overline{\mu}(\beta))} \right) \rho_2^2 + + \frac{g_{21}(\beta)}{2} \quad (35)$$

Considering

$$I_0 = \mathrm{Re}(c_1(0) e^{-i\theta(0)})$$

and $\theta(0)$ given by (24), if $I_0 < 0$ then in the neighborhood of the fixed point (y_0, k_0) there is a stable limit cycle (in a stable invariant closed curve).

Using the formulas from Proposition 5, using the Maple11, for fixed values of p, q, r ($p = 0.4, q = 0.5, r = 0.4, u = 8$, $f(x) = 0.5 * arctan(x)$), we obtained $s_0 = 2.404$ (value of bifurcation), $I_0 = -1,946$ (supercritical bifurcation) and for $n = 1500$ and $\beta = 0.001$,we obtain the following maps (see Fig. 1-5).

Fig.2.(n,Kn)

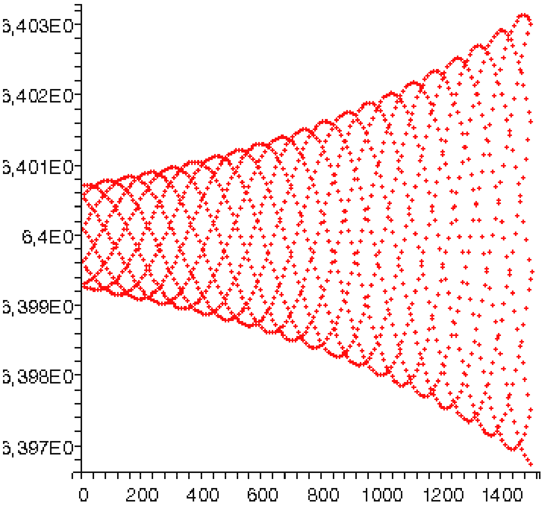


Fig.3.(Yn,Kn)

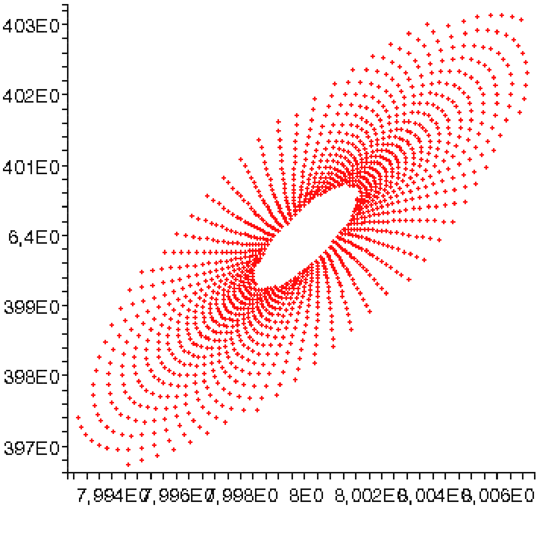


Fig.4.(Un,Kn)

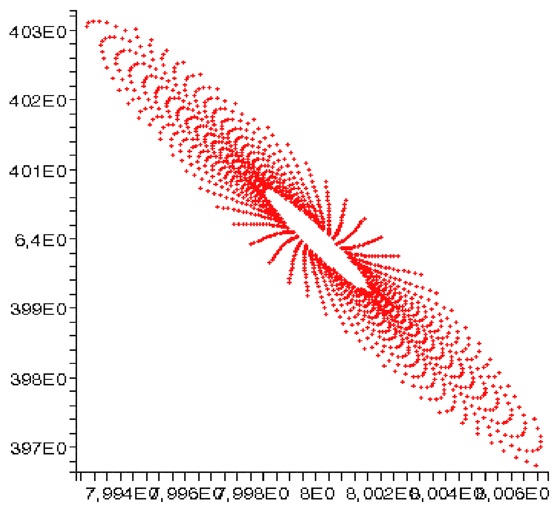
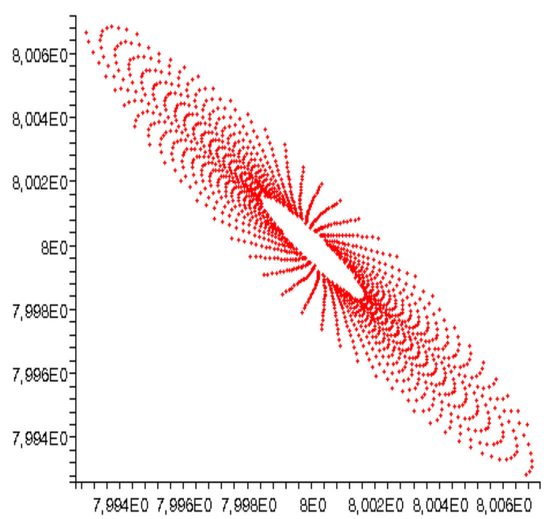


Fig 5. (Un,Yn)



6. CONCLUSIONS

The article focuses on the business cycle model developed by Kaldor, with discrete-time and delay-argument characteristics. In this context, we considered the fact that the variation of capital depends on the value of $(n-m)-$moment income, with $n, m \in \mathbb{N}$ and $m \geq 0$. As mentioned, if $m = 0$, the model represents the discrete–time Kaldor model. On the other hand, if $m = 1$, the Kaldor model corresponds to a dynamic system with discrete–time and delayed – argument. Considering s the adjustment parameter, we were able to determined the value s_0 for which the characteristic equation related to the model in the equilibrium point has unit absolute value complex roots, if $m = 0$ and complex roots with absolute value less than 1, if $m = 1$. By employing the normal forms technique, we found the equation which defines the stable limit cycle related to the model. Further, we used Maple11 to visualize the orbits of the variables of the model. Consequently, the present work establishes the fact that for certain parameters values, there exist a business cycle and allow us to establish the behavior of state variables on different moments. The case of $m \geq 2$ will be considered in a future work since the analysis of this case requires a more laborious research.

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APPENDIX A: THE DISCRETE-DELAY KALDOR MODEL

This section presents the discrete - delay Kaldor model of business cycle for the state variables of income (national income) Y_n and capital stock K_n, where $n \in \mathbb{N}$. This model corresponds to a system of equations with discrete time and delay, given by:

$$\begin{aligned} Y_{n+1} &= Y_n + s\,[I(Y_n, K_n) - S(Y_n, K_n)] \\ K_{n+1} &= K_n + I(Y_{n-m}, K_n) - qK_n \end{aligned} \tag{36}$$

Where: Y_{n-m} represents income on moment $n-m$, with $m \geq 1$, $I : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the investment function and $S : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the savings function, both considered being differentiable functions; s represents an adjustment parameter, which describes the reaction of the system to the difference between investment and saving.

Assuming Keynes's hypothesis which states that the saving function is proportional with income, the proportionality parameter being p, the saving propensity, and also that investment function I is defined à la Rodano with respect to a certain normal level of income u, and a normal level of capital stock - $\frac{pu}{q}$, where $u > 0$, $q \in (0, 1)$,

$$I(Y,K) = pu + r\left(\frac{pu}{q} - K\right) + f(Y-u) \tag{37}$$

where: $r > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) = 0$, $f'(0) \neq 0$ and $f'''(0) \neq 0$, the dynamic system (1), can be written as:

$$\begin{aligned} Y_{n+1} &= (1-sp)Y_n - rsK_n + sf(Y_n - u) + spu\left(1 + \frac{r}{q}\right) \\ K_{n+1} &= (1-s-q)K_n + f(Y_{n-m} - u) + pu\left(1 + \frac{r}{q}\right) \end{aligned} \tag{38}$$

with $m = 0$ and $f(x) = \arctan(x)$ giving the version of the model analyzed by Bischi et all. [1].

Changing the variable with

$$x^1 = Y_{n-m}, ...x^m = Y_{n-1}, x^{m+1} = Y_n, x^{m+2} = K_n$$

the vector associated to system (3) is:

$$\begin{pmatrix} x^1 \\ ... \\ x^m \\ x^{m+1} \\ x^{m+2} \end{pmatrix} \to \begin{pmatrix} x^2 \\ ... \\ x^{m+1} \\ (1-sp)x^{m+1} - rsx^{m+2} + sf(x^{m+1} - u) + spu\left(1 + \frac{r}{q}\right) \\ (1-r-q)x^{m+2} + f(x^1 - u) + pu\left(1 + \frac{r}{q}\right) \end{pmatrix} \tag{39}$$

The fixed points are then defined as the points of coordinates $(y_0, ...y_0, k_0) \in \mathbb{R}^{m+2}$ with (y_0, k_0) solution of the system :

$$\begin{aligned} py + rk - f(y-u) - pu\left(1 + \frac{r}{q}\right) &= 0 \\ (r+q)k - f(y-u) - pu\left(1 + \frac{r}{q}\right) &= 0 \end{aligned} \tag{40}$$

The following statements take place:

We note:

$$\begin{aligned} \rho_1 &= f'(y_0 - u), \rho_2 = f''(y_0 - u), \rho_3 = f'''(y_0 - u) \\ a_{10} &= s(\rho_1 - p), a_{01} = -rs, b_{10} = \rho_1, b_{01} = -q - r \end{aligned} \tag{41}$$

Proposition 6. *(i). The Jacobian matrix of application (4) in the fix point* $(y_0, ...y_0, \mathbf{k}_0)^T$ *is the following:*

$$A = \begin{pmatrix} 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & ... & 1 & 0 \\ 0 & ... & ... & 1 + a_{10} & a_{01} \\ b_{10} & ... & ... & 0 & 1 + b_{01} \end{pmatrix} \tag{42}$$

(ii). The characteristic equation of matrix A is:

$$\lambda^{m+2} - a(s)\lambda^{m+1} + b(s)\lambda^{m} - c(s) = 0 \tag{43}$$

where $a(s) = 2 + a_{10} + b_{01},\ b(s) = (1 + a_{10})(1 + b_{01}),\ c(s) = a_{01}b_{10}$

(iii). The eigenvector $q \in \mathbb{R}^{m+2}$*, that corresponds to the eigenvalue* μ *of matrix* A *has the components:*

$$q_i = \mu^{i-1}, i = 2, .., m+1, q_{m+2} = \frac{b_{10}}{\mu - 1 - b_{01}} \tag{44}$$

The eigenvector $p \in \mathbb{R}^{m+2}$*, that corresponds to the eigenvalue* $\overline{\mu}$ *of matrix* A^T *has the components:*

$$p_1 = \frac{(\overline{\mu} - 1 - a_{10})(\overline{\mu} - 1 - b_{10})}{m(\overline{\mu} - 1 - a_{10})(\overline{\mu} - 1 - b_{01}) + \overline{\mu}(2\overline{\mu} - 2 - a_{10} - b_{01})} \tag{45}$$

$p_i = \frac{1}{\overline{\mu}^{i-1}} p_1,\ \ i = 2, .., m-2, \quad p_{m+1} = \frac{1}{\overline{\mu}^{m-1}(\overline{\mu} - 1 - a_{10})} p_1, \quad p_{m+2} = \frac{\overline{\mu}}{b_{10}} p_1$

The vectors q, p *given by (8) and (9) satisfy the relationship*

$$\sum_{i=1}^{m+2} q_i \overline{p_i} = 1$$