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STRATEGIC DIVIDE AND CHOOSE

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Strategic Divide and Choose[1]

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Abstract

We consider the classic cake-divison problem when the cake is a heterogeneous good represented by an interval in the real line. We provide a mechanism to implement, in an anonymous way, an envy-free and efficient allocation when agents have private information on their preferences. The mechanism is a multi-step sequential game form in which each agent at each step receives a morsel of the cake that is the intersection of what she asks for herself and what the other agent concedes to her.

1 Introduction

Thomson (1996) pointed out that an allocation rule is conceptually different from its selections and the normative properties of its outcomes do not coincide with those of the rule itself. This observation generates two basic questions about fairness: should we focus exclusively on the set of allocations in order to determine criteria for fairness or should we also look at the procedure through which the final outcome is obtained? Should we take the view of procedure fairness or the view of "end-of-state" fairness, or both?

The classic problem of dividing a heterogeneous good (a cake) between two agents offers a great opportunity to analyze these questions in a simple framework. As already noted by Crawford (1977), and previously by Kolm (1972), the classic divide and choose procedure provides an efficient and envy free outcome, but it is hardly considered “fair” when there is complete information on agents’ preferences. The divide and choose procedure provides a no-envy outcome but the procedure itself is not envy free: the chooser envies the role of the divider. To put it in a slightly stronger term, agents are not treated symmetrically in the divide and choose procedure. Fairness can be translated in requirements like anonymity, which is directed to guarantee an ex-ante symmetric treatment of the agents, or like no-envy, which demands an ex-post symmetry among the actual allocations of the agents. It is quite obvious that an allocation rule can satisfy some of these requirements while violating others. Keeping with our simple problem, a mechanism which assigns the entire cake to an agent flipping a (fair!) coin, satisfies anonymity (or procedural no-envy) but clearly violates the (“end-of-state”) no-envy criterion, while the divide and choose rule when the divider is fixed, satisfies no-envy but violates anonymity.

Reconciling efficiency, procedural fairness, and “end-of-state” fairness is not so simple as it could appear at first glance. For instance, one could think that the divide and choose procedure when the divider is randomly chosen is the (simplest) way to make the procedure “fair”. Nevertheless, introducing a random element in the mechanism has many consequences. The set of alternatives over which agents’ preferences are defined is now a set of lotteries. The random mechanism which assigns with equal probability to both agents the role of the divider is equivalent to the lottery which assigns to each agent with equal probability one the following two envy-free allocations: the allocation such that agent 1 is indifferent over the two portions and the allocation such that agent 2 is indifferent over the two portions. Therefore, we need to make the assumption on how agents evaluate lotteries and the normative content of any proposed mechanism will in general be sensitive to the different assumptions. Moreover, even if we assume standard preferences over lotteries, representable by Von Neumann-Morgenstern utility functions, the simple rule which randomly selects the divider may open the door to inefficiency when agents differ in the degree of risk aversion.

In this paper, we focus on the fair division problem when the good to be divided is representable by a linear segment of length one and agents’ preferences are such that single-cut divisions are efficient. Many problems, such as time sharing problems, belong to this class. Consider, for instance, two security guards deciding their shifts during the night: if their preferences depend not only on the number of working hours, but also on their schedule, then the good to be divided (the night hours) is heterogeneous and it can be fair to have shifts of different length; nevertheless it turns to be efficient to divide the night in no more than two shifts, one for each guard. Other examples are related to classic Hotelling models: two ice-cream pedlars have to decide how to partition a beach in two selling regions which can be of different length, since density of bathers may vary along the beach. Again, in order to minimize the pedlars' effort in commuting, it is efficient to partition the beach in two intervals, one for each pedlar.

We propose a normative property which identifies one allocation among those which are envy-free, and provide an anonymous deterministic mechanism which implements it. Our mechanism is a sequential multi-step version of the divide-and-choose mechanism. The assumption that single-cut partitions of the cake are efficient, allows to compare our mechanism with the classic divide and choose only on the ground of fairness.

Let's consider the following example. Two kids, Hansel and Gretel, have to divide a rectangular cake which can be represented as the interval $[0, 1]$. The cake is partly of white chocolate, the interval $[0, m]$, and partly of dark chocolate, the interval $(m, 1]$. Suppose that Hansel prefers the dark chocolate and Gretel the white one, but they are both greedy and to any portion prefer a bigger portion that contains it. Note that any single cut partition of the cake which assigns the left portion to Gretel and the right portion to Hansel is efficient. The problem is where to put the knife in order to be fair. There exists an interval of single-cut points, each of them generating an envy-free and efficient allocation with different utility levels to the greedy kids. The divide and choose procedure where either Hansel or Gretel is the divider implements among the efficient and no-envy allocations the one preferred by the divider. In order to avoid noisy discussions on who is the divider, their mother would help the kids by providing them a way to select an envy-free and efficient allocation in an anonymous way. Uniqueness is relevant in our problem because we cannot leave the kids to choose one allocation in a set of possible solutions, if we really want to avoid noisy discussions. To describe the mechanism in a very intuitive way suppose that the mother knows that Hansel prefers the dark chocolate and Gretel prefers the white. She knows that, once she decides where to put the knife, it is efficient to give the left portion to Gretel and the right portion to Hansel. Unfortunately, she does not know how strong the kids' preferences are over the two types of chocolate. Therefore, she let them choose how to cut the cake. In fact, she proposes the following cake-cutting mechanism to the kids.

Gretel proposes to Hansel to cut the cake at $x_1 \in [0, 1]$. By proposing a single-cut point at x_1, Gretel implicitely asks for herself the portion $[0, x_1]$ and concedes to Hansel the portion $[x_1, 1]$. Hansel may take either the portion $[0, x_1]$ or the portion $[x_1, 1]$. If he takes one of the two portions, then Gretel takes the other portion and the game ends (all the cake has been assigned). If Hensel does not take any portion, then he has to propose a different cut at y_1 with $y_1 < x_1$.

By proposing a single-cut point at y_1, Hansel implicitely asks for himself the portion $[y_1, 1]$ and concedes to Gretel the portion $[0, y_1]$.

Gretel can now choose to take one of the two portions induced by Hansel's cut, that is either the portion $[0, y_1]$ or the portion $[y_1, 1]$. If she takes one of the two portions, then Hansel takes the other portion and the game ends. If Gretel does not take anything, then she receives the portion $[0, y_1]$ and Hansel the portion $[x_1, 1]$. That is, each kid receives the morsel which is the intersection between the portion she wants for herself and what the other kid concedes to her. The interval $[y_1, x_1]$ has still to be assigned and the mechanism is iterated following the same rules until one of the two kids takes one of the portion proposed by the other kid.

The mechanism we propose can be interpreted as a step-by-step negotiation procedure in which agents reach partial agreements. Whenever both agents agree that some part (subset) of the cake should be consumed by one agent, then they accept to assign this part to her. In this way they "reduce" the object over which they dispute and therefore they can more easily reach a definitive agreement.

From a normative point of view, should any kid complains with her mother? Should Hansel pretend to be the first to choose or Gretel the second one? The answer is "no". No matter who moves first, the mechanism implements the same equilibrium allocation. The procedure anonymously selects a no-envy and efficient allocation which has the following characteristics. Consider a subgame starting at any stage t of the dividing game and let $[a, b]$ denote the cake still to be divided. In equilibrium, each agent receives at the current stage t a morsel which has the same value as the overall portion that the other agent receives. Let $([a, S], (S, b])$ be the (efficient) subgame perfect equilibrium allocation, where $[a, S]$ is Gretel's portion. Let x_t and y_t be respectively Gretel and Hansel's proposals at the current stage t according to the subgame perfect equilibrium. Then, Gretel is indifferent between the morsel $[a, y_t]$, the morsel she receives at stage t, and the portion $(S, b]$, i.e. the portion that Hansel consumes in the subgame perfect equilibrium allocation. Similarly, Hansel is indifferent between the morsel $[x_t, b]$, the morsel he receives at stage t, and the portion $[a, S]$. Therefore in each stage t of the game, both agents receive the minimal no-envy morsel, which makes each of them indifferent with respect to the overall portion that the other agent receives in the SPNE allocation of the subgame starting at stage t.

The mechanism described above is slightly more complicated in the general case when the arbitrator does not know how to efficiently divide the cake. But the logic of the mechanism is the same. At each stage agents sequentially propose a partition of the cake and, in case no agent takes one of the portions of the allocation proposed by the counterpart, each of them receives the intersection between what she asks for herself and what the other agent concedes her to consume.

Fair division of an heterogeneous good has been widely analyzed both in the mathematical literature and, more recently, in the economic one; see, respectively, Brams and Taylor (1996) and Robertson and Webb (1998) for two recent books on cake-cutting. Mathematicians have devoted great efforts in order to find the minimal number of cuts needed to fairly cut a "cake" according to the number of eaters (Lester and Spanier (1961), Stromquist (1980), Barbanel and Brams (2001)), while economists have been more interested in generalizing the problem, allowing for larger domains of preferences (Berliant, Dunz and Thomson (1992)), analyzing the case of indivisible goods (Crawford and Heller (1979), Demko and Hill (1988), Alkan, Demange and Gale (1991), Brams and Fishburn (2000), Edelman and Fishburn (2001)), or imposing additional properties, such as consistency or monotonicity requirements (Thomson (1994a), (1994b) Maniquet and Sprumont (2000), among others). In most of these contributions great attention has been devoted to the existence and the axiomatic characterization of a normative solution to this fair division problem, but much less attention to a strategic approach. A relevant exception is the recent paper by Thomson (2005), who showed a simple game form called "divide and permute" to fully implement in Nash equilibrium the no-envy solution in n-person fair division problem. In this paper we follow a similar approach by proposing a game form of a two-agent fair division problem to implement in subgame perfect equilibrium an envy-free and efficient solution in an anonymous way by means of a deterministic mechanism.

2 Notation and Definition

Our model is a simple version of the classical cake division problem. There is a measurable space $(\ ,\mathcal{F})$, where $\ \equiv [0,1]$ (a cake) is the object to be divided between the two agents that can be represented by an interval in the real line, and $\mathcal{F}$ is a σ-algebra over . We say that an element of $\mathcal{F}$ is a *portion* and that an $\mathcal{F}$-measurable subset of a portion is a morsel. Agents have preferences over portions of . Each agent i is endowed with a utility function $u_i : \mathcal{F} \to R^+$ that is a nonatomic probability measure on $\mathcal{F}$.[1] (Since preferences are invariant up to a positive rescaling of the utility function, $u_i(\ ) = 1$ is only a normalization). In particular we assume the following. For both $i = 1, 2$, let $v_i > 0$ be a continous function on $[0,1]$. Agent i's utility of the portion P_i is $u_i(P_i) = \int_{P_i} v_i(s)ds$. When the portion of agent i is an interval, we identify this portion by means of the two extremes of the interval, that is if $P_i \equiv [a,b] \subseteq [0,1]$, we write $u_i(P_i) = u_i(a,b) = \int_a^b v_i(s)ds$. Let U_i be the set of agent i's utility functions and $U = (U_1, U_2)$ be the set of all utility profiles.

An (ordered) partition $P = (P_1, \ldots, P_k)$ of   constituted only by portions is called a portioned k-partition. An *allocation* $P = (P_1, P_2)$ is a portioned two-partition, where P_i is the portion assigned to agent $i = 1, 2$. An allocation $P = (P_1, P_2)$ is *efficient* (or *weakly efficient*, respectively) at $u \in U$ if there exists no other allocation $P' = (P'_1, P'_2)$ such that $u_i(P_i) \geq u_i(P'_i)$ for all i, with the strict inequality holding for some i (or $u_i(P_i) > u_i(P_i)$ for all i). Any

[1] A measure u_i is *nonatomic* if, for each partion A and each x in $(0, u(A))$, there exists another portion $B \subseteq A$ such that $u_i(B) = x$.

efficient allocation is also weakly efficient.[2] An allocation P is *envy-free* at $u \in U$ (or satisfies *no-envy*), if $u_i(P_i) \geq u_i(P_j)$ for $i = 1, 2$.Let $\mathbb{P}$ denote the set of allocations.

An *allocation rule* is a function $f : U \to \mathbb{P}$. Let $f_i(u)$ be the portion assigned by the allocation rule f to agent $i = 1, 2$ at $u \in U$. An allocation rule f is *envy-free* if $f(u)$ is envy-free at every $u \in U$. An allocation rule f is *efficient* if $f(u)$ is efficient at every $u \in U$. An allocation rule is anonymous if interchanging the preferences means interchanging the assigned portion, that is for any $(u_1, u_2) \in U$, if $f_i(u_1, u_2) = P_i$ then $f_i(u_2, u_1) = P_j$ for both $i, j = 1, 2$, $i \neq j$.

In the paper we concentrate on multi-stage sequential mechanisms. Let $\mathbb{Z}_+$ be the set of positive integers. Let ^{t} be the amount of the heterogeneous good still to be divided at stage $t = \{1, 2, 3..., T\}$ with $T \in \mathbb{Z}_+$. By assumption ${}^{1} \equiv [0, 1]$. Let P_i^t denote the morsel of the good that agent $i = 1, 2$ receives at stage t and P_j^t the morsel that the other agent receives at the same stage t. Hence, $P_1^t \cup P_2^t \cup {}^{t+1} = {}^{t}$. We call P_i^t agent i's *current morsel* at stage t. For any $t \leq T$, let $\mathbf{P}_i^t = \cup_{k=t}^{T} P_i^k$ denote the overall portion that agent i's will receive playing the mechanism from stage t onwards; therefore $\mathbf{P}_i^1 = P_i$. We call $\mathbf{P}_i^t$ agent i's *residual portion* at stage t.

Now we are ready to introduce a more demanding property than no-envy. A multi-stage mechanism is *residual-equivalent envy-free* if, at each stage, each agent is indifferent between getting her current morsel and getting the other agent's residual portion. Any residual-equivalent envy-free mechanism not only is envy-free in each stage, but also equalizes the extent to which an agent prefers his own portion to the other agent's portion. Note that although we introduce the concept of residual-equivalent envy-free in the context of multistage mechanisms, it is well defined for any allocation P as in the formal definition below. Let $(P_i^1, ..., P_i^T)$ be a partition of agent i's portion P_i in T morsels.

Definition 1 *An allocation $P = (P_1, P_2)$ is* residual-equivalent envy-free *(REEF) at $u \in U$ if for each $i = 1, 2$ there exists a partition $(P_i^1, ..., P_i^T)$ of P_i such that $u_i(P_i^t) = u_i(\mathbf{P}_j^t)$ for all $t = \{1, 2, ...T\}$ with $T \in \mathbb{Z}_+$. An allocation rule f is residual-equivalent envy-free if $f(u)$ is residual-equivalent envy-free at every $u \in U$.*

We do not provide any strong normative foundation for this requirement. It is a useful tool to prove that the mechanism that we present implements a no-envy and efficient allocation in an anonymous way. In the next section, in fact, we characterize the domain of utility profiles for which any single cut allocation is efficient. Then we show that for any utility profile in this domain, there exists a *unique* allocation that satisfies the above condition. This allocation turns out to be the SPNE allocation of the implementation mechanism we propose.

[2] The converse is true under our assumption that agents have preferences that are mutually absolutely continuous; see Akin (1995, Lemma 9).

3 Existence and Uniqueness of REEF allocations

The classic divide and choose mechanism generates portioned two-partitions. We propose our mechanism as a way of ameliorating it by guaranteeing an anonymous selection of any envy-free and efficient allocation. Hence, we focus on the utility profile domain in which portioned two-partitions are efficient.

(A1) For all $x \in [0,1]$,either the allocation $([0,x],(x,1])$ or $([x,1],[0,x))$ is efficient.

Let U^{sc} denote the domain of utility profiles for which condition A1 holds. Hence, for all utility profiles in U^{sc} and for all $x \in [0,1]$, there always exists an allocation generated by the single-cut x which is efficient. The following Lemma characterizes the set U^{sc}.

Lemma 1 *A sufficient and necessary condition for (A1) is that $v_1(x) - v_2(x)$ is (weakly) monotonic in x.*

Proof. *Sufficiency.* Without loss of generality, suppose $v_1(x) - v_2(x)$ is (weakly) decreasing. Suppose that there is a single-cut partition $([0,a),[a,1])$ which is not efficient, then there exists another partition (P_1, P_2) such that $u_1(P_1) \geq u_1(0,a)$ and $u_2(P_2) \geq u_2(a,1)$ with at least one strict inequality. Because $u_2(P_2) \geq u_2(a,1)$ and $v_2(x) > 0$ by definition, it is impossible that $[0,a) \subset P_1$ such that the (Lebesgue) measure of P_1 is larger than a. Similarly $[a,1] \subset P_2$ is not possible. Let $A = [0,a) \cap P_2$ and $B = [a,1] \cap P_1$. We know that A and B have positive Lebesgue measure. There are three possible scenarios. (1) If $u_1(A) > u_1(B)$, then there exists a set A', $A' \subset A$, such that $u_1(A') = u_1(B)$. Note that $P_1 = (\{[0,a) - A\} \cup B) \subset (\{[0,a) - A'\} \cup B)$. Hence, $u_1(P_1) < u_1(\{[0,a) - A'\} \cup B) = u_1([0,a))$. Contradictory to the claim that (P_1, P_2) is Pareto superior to $([0,a),[a,1])$. (2) If $u_1(A) < u_1(B)$, because $v_1(x) - v_2(x)$ is (weakly) decreasing and A is to the left of B, $u_2(A) < u_2(B)$. There exists a set B', $B' \subset B$, such that $u_2(B') = u_2(A)$. Note that $P_2 = (\{[a,1] - B\} \cup A) \subset (\{[a,1] - B'\} \cup A)$. Hence, $u_2(P_2) < u_2(\{[a,1] - B'\} \cup A) = u_2([a,1])$. Contradiction. (3) If $u_1(A) = u_1(B)$, because $v_1(x) - v_2(x)$ is (weakly) decreasing and A is to the left of B, $u_2(A) \leq u_2(B)$. If $u_2(A) < u_2(B)$, apply the same argument as in (2) and find a contradiction. If $u_2(A) = u_2(B)$, this is contradictory to the claim that (P_1, P_2) is Pareto superior to $([0,a),[a,1])$.

Necessity. Suppose that $v_1(x) - v_2(x)$ is not monotonic in x. Without loss of generality, suppose that there exist three points: a, b, c, with $0 < a < b < c < 1$, such that $v_1(b) - v_2(b) < v_1(c) - v_2(c) < v_1(a) - v_2(a)$[3]. Let y be any point between b and c, i.e., $a < b < y < c$. We now show that both $([0,y),[y,1])$ and $([y,1],[0,y))$ are not efficient allocations. Let's first look at $([0,y),[y,1])$. Intuitively, agent 1 can exchange a tiny slice of the cake centered around b with

[3] If $v_1(x) - v_2(x)$ is not monotonic in x, then either $v_1(x) - v_2(x)$ is U-shaped over certain interval or it is $\cap$-shaped over certain interval. We can find three points: a, b, c, with $0 < a < b < c < 1$, such that either $v_1(b) - v_2(b) < v_1(c) - v_2(c) < v_1(a) - v_2(a)$ or $v_1(a) - v_2(a) < v_1(c) - v_2(c) < v_1(b) - v_2(b)$. The proofs of the two cases are symmetric.

agent 2 for a tiny slice of the cake centered around c, to make both agents better off. Let ϵ_b, ϵ_c be sufficiently small such that for any $x \in [b, b + \epsilon_b]$, $v_1(x) - v_2(x) < v_1(c) - v_2(c)$, where $\epsilon_b < y - b$, and for any $x \in [c, c + \epsilon_c]$, $v_1(x) - v_2(x) > v_1(b) - v_2(b)$; moreover, the following equation holds:

$$- \int_b^{b+\epsilon_b} v_1(x)dx + \int_c^{c+\epsilon_c} v_1(x)dx = 0.$$

We can find such ϵ_b, ϵ_c due to the continuity of agents' utility density functions. By construction, agent 1 is indifferent between $[0, y)$ and $[0, b) \cup [b + \epsilon_b, y) \cup (c, c + \epsilon_c]$; while agent 2 strictly prefers $[y, c] \cup (c + \epsilon_c, 1] \cup [b, b + \epsilon_b)$ to $[y, 1]$, because $\int_b^{b+\epsilon_b} v_2(x)dx - \int_c^{c+\epsilon_c} v_2(x)dx > 0$. Hence, $([0, b) \cup [b + \epsilon_b, y) \cup (c, c + \epsilon_c], [y, c] \cup (c + \epsilon_c, 1] \cup [b, b + \epsilon_b))$ Pareto dominates $([0, y), [y, 1])$. Now let's check $([y, 1], [0, y))$. Similarly, agent 1 can exchange a tiny slice of the cake centered around c with agent 2 for a tiny slice of the cake centered around a, to make both agents better off. The formal proof is omitted. ■

Note that the U^{sc} domain contains the domain of single-peaked (or single-plateaued) utility functions where agents' peaks are on the opposite extremes of the segment $[0, 1]$. For instance, this is a reasonable preference domain in those division problems where the linear interval represents the contested borderland between two countries.

Let $F_i^{[a,b]}$ be the point of the interval $[a, b]$ such that $u_i\left(a, F_i^{[a,b]}\right) = u_i\left(F_i^{[a,b]}, b\right) = \frac{1}{2}u_i(a, b)$. We call $F_i^{[a,b]}$ agent i's indifference point over $[a, b]$. With a little abuse in notation we write $u_i\left(F_i^{[a,b]}\right)$ to denote the utility of agent i in taking one of the two portions, either $\left[a, F_i^{[a,b]}\right]$ or $\left[F_i^{[a,b]}, b\right]$, respectively. We call $u_i\left(F_i^{[a,b]}\right)$ agent i's half-cake-equivalent utility of $[a, b]$.

Lemma 2 *For any preference profile $u \in U^{sc}$, if $F_1^{[0,1]} \leq F_2^{[0,1]}$ $\left(F_1^{[0,1]} \geq F_2^{[0,1]}\right)$ then $F_1^{[a,b]} \leq F_2^{[a,b]}$ $\left(F_1^{[a,b]} \geq F_2^{[a,b]}\right)$ for all $[a, b] \subseteq [0, 1]$.*

Proof: By Lemma 1, $v_1(x) - v_2(x)$ is either weakly increasing or weakly decreasing. If $F_1^{[0,1]} \leq F_2^{[0,1]}$, then $v_1(x) - v_2(x)$ is weakly decreasing. Since $v_1(x) - v_2(x)$ is weakly decreasing over $[0, 1]$, $F_1^{[a,b]} \leq F_2^{[a,b]}$ for all $[a, b] \subseteq [0, 1]$.

Proposition 1 *For any preference profile $u \in U^{sc}$, there exists a unique efficient residual-equivalent envy-free allocation.*

Proof: We provide the intuition of the proof here. The formal proof is in the appendix. For any $u \in U^{sc}$, if $F_1^{[0,1]} = F_2^{[0,1]} = c$, then $([0, c), [c, 1])$ is the unique (in terms of utility) allocation which satisfies REEF property. Without loss of generality, assume $F_1^{[0,1]} > F_2^{[0,1]}$, then in any efficient single cut allocation, agent 1 gets the right part and agent 2 gets the left part of the cake. For any $c \in [0, F_2^{[0,1]})$, $([c, 1], [0, c))$ is not equivalent to an efficient REEF allocation because agent 2 envies agent 1. Similarly, $([c,1],[0,c))$ is not equivalent to an efficient REEF for any $c \in (F_1^{[0,1]}, 1]$. Hence, if $([c,1],[0,c))$ is equivalent to an efficient REEF allocation, then $c \in [F_2^{[0,1]}, F_1^{[0,1]}]$.

For any point $c \in [F_2^{[0,1]}, F_1^{[0,1]}]$, define y_c^1 such that $u_1([0,c)) = u_1([y_c^1, 1])$, and x_c^1 such that $u_2([c,1]) = u_2([0, x_c^1])$. If $([c,1],[0,c))$ is equivalent to an efficient REEF, $(y_c^1, 1]$ is equivalent to agent 1's portion in stage 1, P_1^1, and $[0, x_c^1)$ is equivalent to agent 2's portion in stage 1, P_2^1. Note that x_c^1 and y_c^1 are decreasing and continuous in c. Now we look at agents' division of the remaining cake $[x_c^1, y_c^1]$.

Let $F_1^1 \equiv F_1^{[x_c^1, y_c^1]}$ ($F_2^1 \equiv F_2^{[x_c^1, y_c^1]}$) denote agent 1's (agent 2's) indifferent point over $[x_c^1, y_c^1]$, i.e., $u_1([x_c^1, F_1^1]) = u_1([F_1^1, y_c^1])$. By Lemma 2, $F_1^1 \geq F_2^1$. Similar to the above reasoning, if $([c,1],[0,c))$ is equivalent to an efficient REEF allocation, then c must be in the interval $[F_2^1, F_1^1]$. We can find an interval $[\underline{c}^1, \overline{c}^1]$, which is a strict subset of $[F_2^{[0,1]}, F_1^{[0,1]}]$, such that if $c \notin [\underline{c}^1, \overline{c}^1]$, then $c \notin [F_2^1, F_1^1]$. Hence, if $([c,1],[0,c))$ is equivalent to an efficient REEF allocation, then $c \in [\underline{c}^1, \overline{c}^1]$. If $\underline{c}^1 = \overline{c}^1 = c^*$, we can show that $F_2^1 = F_1^2 = c^*$ and $([0,c^*),[c^*,1])$ is the unique (in terms of utility) efficient REEF allocation. Suppose that $\underline{c}^1 < \overline{c}^1$.

For any $c \in [\underline{c}^1, \overline{c}^1]$, define x_c^2, y_c^2 such that if $([c,1],[0,c))$ is equivalent to an efficient REEF, $(y_c^2, y_c^1]$ is equivalent to agent 1's portion in stage 2 and $[x_c^1, x_c^2)$ is equivalent to agent 2's portion in stage 2, i.e., $(P_1^1, P_2^1) = ((y_c^2, y_c^1], [x_c^1, x_c^2))$. We then look at agents' division of the remaining cake $[x_c^2, y_c^2]$.

Define $x_c^t, y_c^t; F_1^t, F_2^t$, and $\underline{c}^t, \overline{c}^t$ similarly. There are two possible scenarios. (1) At some stage t, $t \in \mathbb{Z}_+$, $\underline{c}^t = \overline{c}^t = c^*$, and then $([0,c^*),[c^*,1])$ is the unique (in terms of utility) efficient REEF allocation. (2) At any stage t, $t \in \mathbb{Z}_+$, $\underline{c}^t < \overline{c}^t$. Since $\lim_{t\to\infty} x_c^t - y_c^t = 0$, then $\lim_{t\to\infty}(\overline{c}^t - \underline{c}^t) = 0$; let $c^* = \lim \underline{c}^t = \lim \overline{c}^t$, then $([c^*,1],[0,c^*))$ is the unique efficient REEF allocation. ■

4 The iterated divide and choose procedure

In this section we present a mechanism to implement the residual-equivalent envy-free allocation in subgame perfect Nash equilibrium. In the introduction we pointed out that when agents have complete information on their counterpart's preferences and behave strategically, the classic divide and choose procedure seems far from being satisfactory from a normative point of view. Hence, this is the case where it is more urgent to find a mechanism which treats agents symmetrically. The mechanism is a multi-stage sequential procedure such that at every stage each agent has the right to propose an allocation, that is a portioned two-partition of the cake specifying which agent should take each portion. If agents propose different allocations, than each agent receives the intersection between what she asks for herself and what the other agent concedes to her.

The mechanism

Any stage $t = \{1,2,...T\}$ with $T \in \mathbb{Z}_+$ is formed by four sequential steps. Let X^t be the allocation proposed by agent 1 at stage t and, with a little abuse in notation, let $x^t \in [0,1]$ denote the single-cut that characterizes this two-portioned partition. Let X_1^t be the portion that agent 1 asks for herself and X_2^t the portion that she concedes to agent 2.[4] Let Y^t be the allocation proposed by agent 2 at stage t, let $y^t \in [0,1]$ denote the associated single-cut and Y_2^t be the portion that agent 2 asks for himself and Y_1^t the portion that he concedes to agent 1.

Stage 1

Step 1 Agent 1 proposes an allocation X^1

Step 2. Agent 2 may take either the portion X_1^1 or the portion X_2^1 or he may propose a different allocation Y^1, such that at least for some $j = 1, 2$, $X_j^1 \cap Y_j^1$ has positive Lebesgue measure.

Step 3: Agent 1 may choose to take either one of the two portions Y_1^1, Y_2^1 or nothing.

Step 4 : If agent 1 does not take any portion and $X_j^1 \cap Y_j^1$ has zero Lebesgue measure for some $j = 1, 2$, then the entire cake is given to the agent $i \neq j$ and the game ends. Otherwhise each agent $i = 1, 2$ receives the morsel $P_i^1 = X_i^1 \cap Y_i^1$.

Note that either the game ends or each agent receives a morsel of the cake of positive size, and the cake that has still to be assigned is an interval.

Consider any stage t and denote by $^t \subset [0,1]$ the cake still to be assigned .

Stage t

Step 1 Agent 1 proposes an allocation $X^t$of the cake ^{t}.

Step 2. Agent 2 may take either the portion X_1^t or the portion X_2^t or he may propose a different allocation Y^t, such that at least for some j, $X_j^t \cap Y_j^t$ has positive Lebesgue measure.

Step 3: Agent 1 may choose to take either one of the two portions Y_1^t, Y_2^t or nothing.

Step 4 : If agent 1 does not take any portion and $X_j^t \cap Y_j^t$ has zero Lebesgue measure for some agent $j = 1, 2$, then the entire cake is given to the agent $i \neq j$ and the game ends. Otherwhise each agent $i \in N$ receives the morsel $P_i^t = X_i^t \cap Y_i^t$.

The mechanism ends at stage T when either one of the agents takes a portion proposed by the counterpart or the entire cake ^{T} has been assigned to some agent.

Proposition 2 *The efficient residual equivalent envy-free allocation is the unique SPNE outcome of the iterated divide and choose mechanism.*

Proof: see the appendix.

Since we proved that the residual equivalent envy-free allocation is unique, it follows that the mechanism is anonymous, as its symmetric structure suggests.

The mechanism might be infinite, and therefore it might be interesting to know if a finite version still has any nice property. Consider a K-truncated version of the mechanism when we exogenously fix the number of iterations, $T = K$, for any finite number K, and at the last stage K agents play the classic divide and choose mechanism (i.e. agent 1 proposes a two-portioned partition

[4] We refer to agent 1 as a female agent and to agent 2 as a male agent

and agent 2 chooses the portion he prefers). Then, the following corollary holds (which follows from Lemma 3 in appendix). Let $T^* \in \mathbb{N}$ be the number of iterations in the SPNE of the non-truncated mechanism.

Corollary 1 *In any $K-$truncated version of the mechanism the SPNE outcome is efficient and envy-free, and*
(i) for all $1 < K \leq T^*$ the utility of both agents is higher than the utility level that the chooser achieves if agents play the divide and choose mechanism;
(ii) for all $K < T^*$ both agents prefer to be agent 1 of the iterated sequential game, but agent 2's utility is increasing in the number of the iterations K.

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5 Appendix

Proposition 1: For any preference profile $u \in U^{sc}$, there exists a unique efficient residual-equivalent envy-free allocation.

Proof. For any $u \in U^{sc}$, if $F_1^{[0,1]} = F_2^{[0,1]} = c$, then $([0,c),[c,1])$ is the unique (in terms of utility) allocation which satisfies REEF property (it is also the unique, in terms of utility, envy-free allocation). Without loss of generality, assume $F_1^{[0,1]} > F_2^{[0,1]}$, then in any efficient single cut allocation, agent 1 gets the right part and agent 2 gets the left part of the cake. For any $c \in [0, F_2^{[0,1]})$, $([c,1],[0,c))$ is not equivalent to an efficient residual-equivalent envy-free allocation because agent 2 envies agent 1. Similarly, $([c,1],[0,c))$ is not equivalent to an efficient REEF for any $c \in (F_1^{[0,1]}, 1]$.

For any $c \in [F_2^{[0,1]}, F_1^{[0,1]}]$, define y_c^1 such that $u_1([0,c)) = u_1([y_c^1,1])$, and x_c^1 such that $u_2([c,1]) = u_2([0,x_c^1])$. If $([c,1],[0,c))$ is equivalent to an efficient REEF, $(y_c^1,1]$ is equivalent to agent 1's portion in stage 1, P_1^1, and $[0,x_c^1)$ is equivalent to agent 2's portion in stage 1, P_2^1. Note that x_c^1 and y_c^1 are decreasing and continuous in c. Let $F_1^{[x_c^1,y_c^1]}$ $(F_2^{[x_c^1,y_c^1]})$ denote agent 1's (agent 2's) indifferent point over $[x_c^1,y_c^1]$, i.e., $u_1([x_c^1, F_1^{[x_c^1,y_c^1]}]) = u_1([F_1^{[x_c^1,y_c^1]}, y_c^1])$.

Claim: $F_1^{[x_c^1,y_c^1]}$ and $F_2^{[x_c^1,y_c^1]}$ are continuous functions of c from $[F_2^{[0,1]}, F_1^{[0,1]}]$ into $[F_2^{[0,1]}, F_1^{[0,1]}]$.

Proof of the Claim: Because $u \in U^{sc}$and$F_1^{[0,1]} > F_2^{[0,1]}$, by Lemma 2, $F_1^{[x_c^1,y_c^1]} \geq F_2^{[x_c^1,y_c^1]}$. Since x_c^1 and y_c^1 are decreasing and continuous in c, $F_1^{[x_c^1,y_c^1]}$ and $F_2^{[x_c^1,y_c^1]}$ are decreasing and continuous in c. (See figure 1 for illustration.)

Let $\overline{F}_1^1$ denote the upper bound of $F_1^{[x_c^1,y_c^1]}$ for $c \in [F_2^{[0,1]}, F_1^{[0,1]}]$. When $c = F_2^{[0,1]}$, $F_1^{[x_c^1,y_c^1]}$ achieves its maximum on $[F_2^{[0,1]}, F_1^{[0,1]}]$. $\overline{F}_1^1 = F_1^{[x^1_{F_2^{[0,1]}}, y^1_{F_2^{[0,1]}}]} = F_1^{[F_2^{[0,1]}, y^1_{F_2^{[0,1]}}]} = F_1^{[0,1]}$. (See figure 2 for illustration). The last equality follows the fact that $u_1([0, F_2^{[0,1]}]) = u_1([y^1_{F_2^{[0,1]}}, 1])$ and $u_1([0, F_1^{[0,1]}]) = u_1([F_1^{[0,1]}, 1])$. Let $\underline{F}_2^1$ denote the lower bound of $F_2^{[x_c^1,y_c^1]}$ for $c \in [F_2^{[0,1]}, F_1^{[0,1]}]$. When $c = F_1^{[0,1]}$, $F_2^{[x_c^1,y_c^1]}$ achieves its minimum on $[F_2^{[0,1]}, F_1^{[0,1]}]$. Similarly, we have $\underline{F}_2^1 = F_2^{[0,1]}$. Since $F_1^{[x_c^1,y_c^1]} \geq F_2^{[x_c^1,y_c^1]}$, we have $F_2^{[0,1]} \leq F_2^{[x_c^1,y_c^1]} \leq F_1^{[x_c^1,y_c^1]} \leq F_1^{[0,1]}$ for any $c \in [F_2^{[0,1]}, F_1^{[0,1]}]$. Therefore, $F_1^{[x_c^1,y_c^1]}$ and $F_2^{[x_c^1,y_c^1]}$ are continuous functions of c from $[F_2^{[0,1]}, F_1^{[0,1]}]$ into itself. ■

(Insert figure 1 and 2 here)

By Brouwer's fixed point theorem, there exists c such that $F_2^{[x_c^1,y_c^1]} = c$. Let $\underline{c}^1$ denote the smallest fixed point such that $F_2^{[x_c^1,y_c^1]} = c$. Note that when $c = F_2^{[0,1]}$, $x_c^1 = F_2^{[0,1]}$ and $F_2^{[x_c^1,y_c^1]} > F_2^{[0,1]}$, so $c = F_2^{[0,1]}$ is not a fixed point of $F_2^{[x_c^1,y_c^1]}$. Therefore $\underline{c}^1 > F_2^{[0,1]}$. Since $F_2^{[x_c^1,y_c^1]}$ is decreasing in c, for any $c \in [F_2^{[0,1]}, \underline{c}^1)$, $c < \underline{c}^1 \leq F_2^{[x_c^1,y_c^1]} \leq F_1^{[x_c^1,y_c^1]}$; hence, $([c,1],[0,c))$ is not an efficient REEF for any $c \in [F_2^{[0,1]}, \underline{c}^1)$. Because if $([c,1],[0,c))$ is an efficient REEF, then $(P_1^1, P_2^1) = ((y_c^1, 1], [0, x_c^1))$, and agent 2 envies agent 1's share over the remaining cake $[x_c^1, y_c^1]$. Similarly, there exists a c such that $F_1^{[x_c^1,y_c^1]} = c$. Let $\overline{c}^1$ denote the largest fixed point such that $F_1^{[x_c^1,y_c^1]} = c$. Note that $c = F_1^{[0,1]}$ is not a fixed point of $F_1^{[x_c^1,y_c^1]}$, therefore $\overline{c}^1 < F_1^{[0,1]}$. Similarly, for any $c \in (\overline{c}^1, F_1^{[0,1]}]$, $c > F_1^{[x_c^1,y_c^1]} \geq F_2^{[x_c^1,y_c^1]}$, therefore, $([c,1],[0,c))$ is not an efficient REEF for any $c \in (\overline{c}^1, F_1^{[0,1]}]$. Since $F_2^{[x_c^1,y_c^1]} \leq F_1^{[x_c^1,y_c^1]}$ for all $c \in [F_2^{[0,1]}, F_1^{[0,1]}]$ and both are decreasing in c, $\underline{c}^1 \leq \overline{c}^1$. We just established that if $([c,1],[0,c))$ is equivalent to an efficient REEF, then $c \in [\underline{c}^1, \overline{c}^1]$.

If $\underline{c}^1 = \overline{c}^1 = c^*$, then by the definition of fixed points, $F_2^{[x_c^1,y_c^1]} = F_1^{[x_c^1,y_c^1]} = c^*$ and the proposition is proved: the allocation $([c^*,1],[0,c^*))$ is the unique (in terms of utility) efficient REEF allocation.

If $\underline{c}^1 < \overline{c}^1$, for any $c \in [\underline{c}^1, \overline{c}^1]$, define x_c^2 such that $u_2([x_c^1, x_c^2]) = u_2([c, y_c^1])$ and y_c^2 such that $u_1([y_c^2, y_c^1]) = u_1([x_c^1, c])$. If $([c,1],[0,c))$ is equivalent to an efficient REEF, $(y_c^2, y_c^1]$ is equivalent to agent 1's portion in stage 2, P_1^2, and $[x_c^1, x_c^2)$ is equivalent to agent 2's portion in stage 2, P_2^2. Let $F_1^{[x_c^2,y_c^2]}$ $(F_2^{[x_c^2,y_c^2]})$ denote agent 1's (agent 2's) indifference point over $[x_c^2, y_c^2]$, i.e., $u_1([x_c^2, F_1^{[x_c^2,y_c^2]}]) =$

$u_1([F_1^{[x_c^2, y_c^2]}, y_c^2])$.

Similar to the proof of above claim, we can establish that $F_1^{[x_c^2, y_c^2]}$ and $F_2^{[x_c^2, y_c^2]}$ are continuous functions of c from $[\underline{c}^1, \overline{c}^1]$ into $[\underline{c}^1, \overline{c}^1]$. Therefore, by Brouwer's fixed point theorem, $F_1^{[x_c^2, y_c^2]}$ and $F_2^{[x_c^2, y_c^2]}$ have fixed points. Let $\underline{c}^2$ denote the smallest fixed point such that $F_2^{[x_c^2, y_c^2]} = c$ and let $\overline{c}^2$ denote the largest fixed point such that $F_1^{[x_c^2, y_c^2]} = c$. When $c = \underline{c}^1$, $F_2^{[x_c^1, y_x^1]} = c$ (by definition of the fixed point) and $x_c^2 = c$, therefore $F_2^{[x_c^2, y_c^2]} > c$. So $c = \underline{c}^1$ is not a fixed point of $F_2^{[x_c^2, y_x^2]}$. Moreover, $\underline{c}^2 \in [\ \underline{c}^1, \overline{c}^1]$, so $\underline{c}^1 < \underline{c}^2$. Similarly we establish $\underline{c}^1 < \underline{c}^2 \leq \overline{c}^2 < \overline{c}^1$.

If $\underline{c}^2 = \overline{c}^2 = c^*$, then the proposition is proved and the allocation $([c^*, 1], [0, c^*))$ is the unique (in terms of utility) efficient REEF allocation. If $\underline{c}^2 < \overline{c}^2$, for any $c \in [\underline{c}^2, \overline{c}^2]$, define x_c^3, y_c^3, $F_1^{[x_c^3, y_c^3]}$, $F_2^{[x_c^3, y_c^3]}$, and $\underline{c}^3, \overline{c}^3$ similarly. Hence, for any $c \in [\underline{c}^t, \overline{c}^t]$, define $x_c^{t+1}, y_c^{t+1}; F_1^{[x_c^{t+1}, y_c^{t+1}]}$, $F_2^{[x_c^{t+1}, y_c^{t+1}]}$, and $\underline{c}^{t+1}, \overline{c}^{t+1}$ similarly. By definition $\underline{c}^t < \underline{c}^{t+1} \leq \overline{c}^{t+1} < \overline{c}^t$ for all t. Also by definition, for any $c \in [\underline{c}^t, \overline{c}^t]$, $x_c^t < \underline{c}^t < \overline{c}^t < y_c^t$. If for any $t < \infty$, $\underline{c}^t = \overline{c}^t = c^*$, then the proposition is proved and the allocation $([c^*, 1], [0, c^*))$ is the unique (in terms of utility) efficient REEF allocation. If $\underline{c}^t < \overline{c}^t$ for all $t < \infty$, and $\lim \underline{c}^t < \lim \overline{c}^t$, then for any $c \in (\lim \underline{c}^t, \lim \overline{c}^t)$, $\lim x_c^t < c < \lim y_c^t$, which implies that $u_1([\lim x_c^t, c]) = 0$. It is contradictory to $v_1 > 0$. Therefore $\lim \underline{c}^t = \lim \overline{c}^t$. Let $c^* = \lim \underline{c}^t = \lim \overline{c}^t$. It is straightforward that $([c^*, 1], [0, c^*))$ is the unique (in terms of utility) efficient REEF allocation when $F_2^{[0,1]} < F_1^{[0,1]}$. ■

Proof of Proposition 2: To prove this proposition we proceed by proving some easy lemmata. Let $a_t = \min\{x_{t-1}, y_{t-1}\}$ and $b_t = \max\{x_{t-1}, y_{t-1}\}$ for all $t > 1$ and $a^1 = 0$ and $b^1 = 1$. We assume that agents only use stationary strategies in the sense that at each stage t agents' strategies only depend upon the cake still to be divided, $\ ^t = [a_t, b_t]$, and on the proposals made at this stage. From now on we suppose, without loss of generality, that $F_1^{[a_t, b_t]} \leq F_2^{[a_t, b_t]}$.

Lemma 3 *Consider any subgame starting at step 1 of some stage t. Let $\mathbf{P}_i^t$ denote agent i' s portion in the subgame perfect equilibrium allocation of the subgame $\ ^t$. Then, $u_i(\mathbf{P}_i^t) \geq u_i\left(F_i^{[a_t, b_t]}\right) = \frac{1}{2} u_i(\ ^t)$, for both $i = 1, 2$, and for all $t = \{1, 2, ..., T\}$.*

Proof. We actually prove a stronger claim, that is for all $t = \{1, 2, ..., T\}$ each agent has a strategy that guarantees her to obtain $u_i(\mathbf{P}_i^t) \geq u_i\left(F_i^{[a_t, b_t]}\right)$ (not only in equilibrium). Let X^t be the allocation proposed by agent 1 at stage t and $\ ^t \subset [0, 1]$ the cake still to be assigned at this stage. Either $u_2(X_1^t) \geq \frac{u_2(-\ ^t)}{2}$ or $u_2(X_2^t) > \frac{u_2(-\ ^t)}{2}$. Hence, agent 2 by taking his preferred portion obtains $u_2(\mathbf{P}_2^t) \geq \frac{u_2(-\ ^t)}{2}$. Agent 1 can also guarantee herself at least her half-cake-equivalent utility of stage t. Suppose that agent 1 announces $x_t = F_1^{[a_t, b_t]}$ and proposes for herself the portion $[a_t, x_t]$. Either agent 2 takes one of the two portions, and then the claim is proved, or he announces an allocation Y^t and

the proof is completed noticing that either $u_1(Y_1^t) > \frac{u_1(- ^t)}{2}$ or $u_1(Y_2^t) > \frac{u_1(- ^t)}{2}$.
■

Lemma 4 *In all subgame perfect equilibria of the game the mechanism ends only if at stage $T \in \mathbb{Z}_+$, agent 2 chooses one morsel of the allocation proposed by agent 1 and both morsels of the allocation are indifferent for agent 2.*

Proof. Consider any stage t of the game. By design, the mechanism ends either if one of the agent takes one portion of the allocation proposed by the other agent, or if for some $j \in N$ $X_j^t \cap Y_j^t = \emptyset$. In this last case there exists one agent who receives a morsel of zero Lebesgue measure contradicting Lemma 3. Now we prove the following two claims.

Claim 1: *Agent 1 never chooses to end the game at step 3.* Suppose that $X_1^t = [a_t, x_t]$. We already proved that in equilibrium $Y_1^t = [a_t, y_t]$ (otherwhise there exists at least one agent j for which $X_j^t \cap Y_j^t = \emptyset$). There are two cases: (i) $y_t > x_t$. In this case agent 1's best response is to take the portion $[a_t, y_t]$. But then agent 2's best response at step 2 cannot be to propose Y^t, because taking $[x_t, b_t]$ he would obtain a higher payoff. (ii) $y_t < x_t$. We suppose that agent 1 takes one of the two morsels and we show that this cannot occurs along the equilibrium path. If agent 1 takes the morsel $[y_t, b_t]$, then to announce y_t cannot be a best response for agent 2 since he could obtain a morsel $[a_t, x_t]$ at step 2 which contains the morsel $[a, y^t]$. If agent 1 takes the morsel $[a^t, y^t]$, she could obtain a greater utility by not taking any morsel, in which case agent 1 receives the morsel $[a^t, y^t]$ at this stage and some morsel with positive Lebesgue measure in the ensuing stages (by Lemma 3). The proof of the case $X_1^t = [x_t, b_t]$ follows the same argument.

Claim 2: *Agent 2 chooses to end the game only if agent 1 partitions the cake in two portions which are indifferent for agent 2.* Let x_t denote the cut-point proposed by agent 1. If agent 2 takes a portion which is strictly preferred by him to the other portion, then it must be that either $x_t > F_2^{[a_t,b_t]}$ and he takes the portion $[a_t, x_t]$ or $x_t < F_2^{[a_t,b_t]}$ and he takes the portion $[x_t, b_t]$. The first case contradicts Lemma 3 since agent 1 receives less than her half-cake-equivalent utility . Hence it must be that $F_1^{[a_t,b_t]} \leq x_t < F_2^{[a_t,b_t]}$. If agent 2's best response is to take the morsel $[x_t, b_t]$, then it must be the case that $F_1^{[a_t,b_t]} = x_t$. Suppose not and let $F_1^{[a_t,b_t]} < x_t$. By proposing $Y_1^t = [a_t, y_t]$ with $F_1^{[a_t,b_t]} < y_t < x_t$ agent 2 will obtain a higher payoff. In fact either agent 1 takes the morsel $[a_t, y_t]$ or she does not take anything. But then agent 2 either will receive the entire cake (if $X_1^t = [x_t, b_t]$) or he will receive the morsel $[x_t, b_t]$ at stage t and some morsel with positive Lebesgue measure in the ensuing stages. It follows that if agent 2 takes his strictly preferred portion $[x_t, b_t]$, then it must be that $F_1^{[a_t,b_t]} = x_t < F_2^{[a_t,b_t]}$. We now prove that to propose $X_1^t = [a_t, x_t]$ is not a best respose for agent 1. By continuity there exists a different proposal $\tilde{X}_1^t = [a_t, \tilde{x}_t]$ with $x_t < \tilde{x}_t < F_2^{[a_t,b_t]}$ such that either agent 2 picks up $[\tilde{x}_t, b_t]$ (and therefore agent 1 receives the portion $[a_t, \tilde{x}_t] \supset [a_t, x_t]$), or he proposes a different single-cut at y_t. If $y_t = x_t$ then agent 1 by accepting the proposal

receives the morsel $[a_t, y_t] = [a_t, x_t]$ at stage t and some morsels with positive Lebesgue measure in the ensuing stages. If $y_t \neq x_t$ then there exists a morsel that agent 1 can take whose value is strictly higher than $[a_t, x_t] = \left[a_t, F_1^{[a_t,b_t]}\right]$.
■

Now we can easily prove the following.

Lemma 5 *In any subgame perfect equilibrium path, the mechanism ends at stage $T \in \mathbb{Z}_+$ if and only if there exists a unique envy free allocation.*

Proof. *Necessity*: We have already proved that the mechanism ends at stage t only if $x_t = F_2^{[a_t,b_t]}$. Suppose that agent 1 proposes $X_1^t = [a_t, x_t]$. We prove that there exists a proposal y_t which gives agent 2 a higher payoff than the half-cake utility level. Suppose, in fact, that agent 2 proposes $Y_1^t = [a_t, y_t]$ with $F_1^{[a_t,b_t]} < y_t < F_2^{[a_t,b_t]}$. If agent 1 takes a portion, she will take the portion $[a_t, y_t]$. If she does not take any portion, agent 2 will receive the morsel $[x_t, b_t]$ at stage t and some morsel with positive Lebesgue measure in the ensuing stages. Suppose now that agent 1 proposes $X_1^t = \left[F_2^{[a_t,b_t]}, b_t\right]$. Then agent 2 will announce $Y_1^t = \left[a_t, F_1^{[a_t,b_t]}\right]$ forcing agent 1 to take this morsel $\left[a_t, F_1^{[a_t,b_t]}\right]$ at stage 3 (otherwise agent 2 receives all the cake). But obviously to propose $X_1^t = \left[F_2^{[a_t,b_t]}, b_t\right]$ is not a best response since, as we just proved, agent 1 obtains a higher payoff by announcing $X_1^t = [a_t, x_t]$.

Sufficiency: Suppose now that at some $T \in \mathbb{Z}_+$ it exists a unique envy free allocation, characterized by the point $z = F_1^{[a_t,b_t]} = F_2^{[a_t,b_t]}$. The result direcly follows from Lemma 3. ■

Lemma 6 *In all subgame perfect equilibria $X_1^t = [a_t, x_t]$ and $Y_1^t = [a_t, y_t]$ with $x_t > y_t$ for all $t \neq T$.*

Proof. Consider any stage $t < T$. By Lemma 2 $F_1^{[a_t,b_t]} < F_2^{[a_t,b_t]}$ for all $t < T$. Suppose $X_1^t = [x_t, b_t]$. If $x_t \leq F_1^{[a_t,b_t]}$ then agent 2 will pick the portion $[x_t, b_t]$, contradicting Lemma 5. If $x_t > F_1^{[a_t,b_t]}$ then agent 2 can propose $Y_1^t = \left[a_t, F_1^{[a_t,b_t]}\right]$ forcing agent 1 to take the portion $\left[a_t, F_1^{[a_t,b_t]}\right]$ at step 3 (note that $\cap_{i \in N} P_1^i(\ \ ^t) = \emptyset$). But by proposing $X_1^t = \left[a_t, F_2^{[a_t,b_t]}\right]$ agent 1 can obtain a payoff strictly higher than the half-cake utility level. Hence$X_1^t = [a_t, x_t]$ with $x_t > F_1^{[a_t,b_t]}$. By Lemma 5 agent 2 makes a proposal that induces agent 1 to not take any morsel and by Lemma 3 both agents receive a morsel that positive Lebesgue measure. Then it must be that $Y_1^t = [a_t, y_t]$. Finally note that if $y_t > x_t$ agent 2 is not playing a best response since he would obtain a higher payoff by picking up the portion $[x_t, b_t]$. Hence $P_1^2(\ \ ^t) = [a_t, y_t]$ with $y_t < x_t$.[5]
■

[5]It is straightforward to note that if $F_1^{[a_t,b_t]} > F_2^{[a_t,b_t]}$then $X_1^t = [x_t, b_t]$ and $Y_1^t = [y_t, b_t]$ with $x_t < y_t$ for all $t \neq T$.

Corollary 2 *If there exists a subgame perfect equilibrium outcome of the game, then it is efficient.*

Since the SPNE outcome is efficient, and the game is a sequential game with perfect information, if there exists a subgame perfect equilibrium outcome of the game, then it is unique in terms of utility. Suppose, in fact, that there are at least two SPNEs which are not unique in terms of utility. Since the game is of perfect information, then any information set is a singleton. Since the game is also sequential, then there exists at least one player who is indifferent between the two SPNE outcomes, otherwise he is not playing a best response in at least one of his information sets. If one agent is indifferent between the two SPNE equilibria and they are not unique in terms of utility, then the other player strictly prefers one SPNE outcome to the other, and efficiency is violated in at least one case.

It follows that, if it exists, the SPNE outcome of the game is a single cut partition that we denote by (P_1, P_2). Let $S \in [0,1]$ the point which characterizes the SPNE outcome. By Corollary $6P_1 = [0, S] \equiv$ and $P_2 = (S, 1]$. Consider any subgame starting at stage $t \in \{1, ..., T\}$ and let $^t = [a_t, b_t]$ be the cake still to be assigned. Since we assumed $F_1^{[0,1]} \leq F_2^{[0,1]}$, then by Lemma 2, $F_1^{[a_t,b_t]} \leq F_2^{[a_t,b_t]}$ for all $t = 1, 2, ..., T$. By Lemma 3, $S \in [F_1^{[a_t,b_t]}, F_2^{[a_t,b_t]}]$ for all $t = 1, 2, ..., T$.

We now show that the following strategy profile (T1,T2) is a subgame perfect equilibrium:

Let $R_{[a^t,b^t]}$ denote the single-cut point of the unique efficient REEF allocation of the cake $[a_t, b_t]$. As shown in the proof of Proposition 1, $F_1^{[a_t,b_t]} < R_{[a_t,b_t]} < F_2^{[a_t,b_t]}$.

T1: In any $t = 1, 2, ..., T$,
Step 1: agent 1 proposes $X_1^t = [a_t, x_t]$ with $x_t \geq F_2^{[a_t,b_t]}$ such that $u_2(X_2^t) = u_2(a_t, R_{[a^t,b^t]})$;

If agent 2 at Step 2 proposes a different allocation Y^t, then
Step 3:

1. if $X_1^t \cap Y_1^t = \emptyset$ then agent 1 takes her preferred portion in Y^t
2. if $X_2^t \cap Y_2^t = \emptyset$ then agent does not take anything
3. if $X_j^t \cap Y_j^t \neq \emptyset$ for both $j = 1, 2$, and
   (a) $Y_1^t = [a_t, y_t] \subset X_1^t = [a_t, x_t]$, then agent 1 does not take anything if $u_1(Y_1^t) + u_1(y_t, R_{[y_t,x_t]}) \geq u_1(Y_2^t)$, takes the portion Y_2^t if $u_1(Y_1^t) + u_1(y_t, R_{[y_t,x_t]}) < u_1(Y_2^t)$;
   (b) $Y_1^t = [a_t, y_t] \supset X_1^t = [a_t, x_t]$, then agent 1 takes her preferred portion in Y^t;
   (c) $Y_1^t = [y_t, b_t] \subset X_1^t = [x_t, b_t]$ then agent 1 does not take anything if $u_1(Y_1^t) + u_1(x_t, R_{[x_t,y_t]}) \geq u_1(Y_2^t)$, takes the portion Y_2^t otherwise;

(d) $Y_1^t = [y_t, b_t] \supset X_1^t = [x_t, b_t]$ then agent 1 takes her preferred portion in Y^t;

T2: In any $t = 1, 2, ..., T$, agent 2

1. takes $[x_t, b_t]$ if $x_t \leq F_1^{[a_t, b_t]}$;
2. proposes $Y_1^t = \left[a_t, F_1^{[a_t,b_t]}\right]$ if $x_t > F_1^{[a_t,b_t]}$, $X_1^t = [x_t, b_t]$ and $u_2(F_1^{[a_t,b_t]}, b_t) \geq u_2(a_t, x_t)$
3. proposes $Y_1^t = [a_t, y_t]$ such that $u_1(Y_2^t) = u_1(a_t, R_{[y_t,x_t]})$, if $X_1^t = [a_t, x_t]$, $x_t > F_1^{[a^t,b^t]}$ and $u_2(X_1^t) \leq u_2(X_2^t) + u_2\left(R_{[y_t,x_t]}, x_t\right)$;
4. takes $X_1^t = [a_t, x_t]$, otherwise.

The mechanism we propose results in games that are either with finite horizon or continuous at infinity. Therefore, we can apply the one-stage-deviation principle[6], i.e., a strategy profile, s, is subgame perfect if and only if it satisfies the one-stage-condition that no player i can gain by deviating from s in a single stage and conforming to s thereafter.

The following Lemma is the last ingredient we need before proving that the strategy profile (T1,T2) is a SPNE of the game.

Lemma 7 *For any $u \in U^{sc}$, let $R_{[a,b]}$ denote a single-cut point of the REEF allocation for the interval $[a, b]$. It must be true that for any $\tilde{b} < b$, $R_{[a,\tilde{b}]} < R_{[a,b]}$ and for any $\tilde{a} > a$, $R_{[\tilde{a},b]} > R_{[a,b]}$.*

Proof. By definition of the REEF allocation, if $R_{[a,b]}$ is the single-cut point of the REEF allocation, there exists a partition $(P_i^1, ..., P_i^T)$ of P_i such that $u_i(P_i^t) = u_i(\mathbf{P}_j^t)$ for all $t = \{1, 2, ..., T\}$ and for both $i = 1, 2$. We can find a sequence of points $(p_1^1, p_1^2, ..., p_1^T)$, where $a < p_1^1 < p_1^2 < ... < p_1^{T-1} < p_1^T = R_{[a,b]}$, such that $u_1(P_1^1) = u_1([a, p_1^1])$, $u_1(P_1^2) = u_1(p_1^1, p_1^2), ..., u_1\left(P_1^T\right) = u_1(p_1^{T-1}, R_{[a,b]})$. We can also find a sequence of points $(p_2^T, p_2^{T-1}, ..., p_2^1)$, where $R_{[a,b]} = p_2^T < p_2^{T-1} < p_2^{T-2} ... < p_2^1 < b$, such that $u_2(P_2^1) = u_2([p_2^1, b])$, $u_2(P_2^2) = u_2([p_2^2, p_2^1]), ..., u_2(P_2^T) = u_2([R_{[a,b]}, p_2^{T-1},])$. Define $(\tilde{p}_1^Q, ..., \tilde{p}_1^1)$ and $\left(\tilde{p}_2^1, ..., \tilde{p}_2^Q\right)$ similarly for the REEF allocation for the interval $[a, \tilde{b}]$. Suppose that $R_{[a,\tilde{b}]} > R_{[a,b]}$ for some $\tilde{b} < b$. By definition of the REEF allocation, $u_2(p_2^1, b) = u_2(a, R_{[a,b]})$ and $u_2(\tilde{p}_2^1, \tilde{b}) = u_2(a, R_{[a,\tilde{b}]})$. Since $R_{[a,\tilde{b}]} > R_{[a,b]}$ and $\tilde{b} < b$, $\tilde{p}_2^1 < p_2^1$. Moreover, by definition of the REEF allocation, $u_1(a, p_1^1) = u_1(R_{[a,b]}, b)$ and $u_1(a, \tilde{p}_1^1) = u_1(R_{[a,\tilde{b}]}, \tilde{b})$; therefore $\tilde{p}_1^1 < p_1^1$. Similarly, we can show that $\tilde{p}_2^t < p_2^t$ and $\tilde{p}_1^t < p_1^t$ for all $t \leq \min\{T, Q\}$. If $T \leq Q$, $\tilde{p}_2^{Q-1} < p_2^{T-1}$ and $\tilde{p}_1^{Q-1} <$

[6] See Theorem 4.1 and Theorem 4.2 in Fudenberg and Tirole "Game Theory" page 109-110.

p_1^{T-1}. By definition of REEF allocation, $u_2(R_{[a,b]}, p_2^{T-1}) = u_2(p_1^{T-1}, R_{[a,b]})$ and $u_2(R_{[a,\tilde{b}]}, \tilde{p}_2^{Q-1}) = u_2(\tilde{p}_1^{Q-1}, R_{[a,\tilde{b}]})$. If $R_{[a,\tilde{b}]} > R_{[a,b]}$, these two equalities are not compatible. Similarly, we can find contradiction when $T \geq Q$. The proof for the part "for any $\tilde{a} > a$, $S_{[a,\tilde{b}]} > S_{[a,b]}$" follows the same argument. ■

Now we are ready to prove the existence and uniqueness of the SPNE allocation. We first note that if the strategy profile (T1,T2) is a SPNE of the game, then the equilibrium outcome is the allocation $\left([a_t, R_{[a_t,b_t]}), \left[R_{[a_t,b_t]}, b_t\right]\right)$.

Now we prove that T1 is best response to strategy T2 in all subgames.

Consider step 3 of any stage $t < T$. It is easy to check that strategy T1 is best response in cases (1), (2) . Suppose that $X_j^t \cap Y_j^t = \emptyset$ for both $j = 1, 2$.We consider two cases:

(3a) If agent 1 decides to continue the game, then she does not take any portion and,by one-stage deviation principle, obtains a final payoff equal to $u_1(a_t, y_t) + u_1(y_t, R_{[y_t,\tilde{x}_t]})$. If she decides to stop the game, then her best response is clearly to take the portion $[y_t, b_t]$. According to strategy T1 she compares these two payoffs and therefore it is straightforward that she is playing a best response. A similar argument holds for the (3c) case.

(3b) In this case by deviating and not taking anything agent 1 receives the current morsel $[a_t, x_t]$ and some portion of the remaning cake $[x_t, y_t]$. Since by Lemma 3 and by the one-stage deviation principle she does not receive the entire cake $[x_t, y_t]$ in the ensuing stages, then by deviating she receives a final portion that it is strictly contained in the portion $[a_t, y_t]$. Therefore there is no deviation better than the response prescribed in strategy T1. The same argument holds for case (3d)

Consider now step 1 of any stage t. If agent 1 deviates and proposes any allocation with single-cut point $x_t \leq F_1^{[a_t,b_t]}$, then agent 2 takes the portion $[x_t, b_t]$ and therefore agent 1 obtains a payoff equal to $u_1(a_t, x_t) < u_1(a_t, R_{[a_t,b_t]})$. If she proposes $X_1^t = [x_t, b_t]$ with $x_t > F_1^{[a_t,b_t]}$, then agent 2 at step 2 proposes $Y_1^t = \left[a_t, F_1^{[a_t,b_t]}\right]$ and agent 1 obtains a payoff equal to $u_1(a_t, F_1^{[a_t,b_t]}) < u_1(a_t, R_{[a_t,b_t]})$. Finally, if she proposes $X_1^t = [a_t, \tilde{x}_t]$with $\tilde{x}_t < x_t$, by Lemma 7 she cannot obtain a higher payoff, while if she offers $\tilde{x}_t > x_t \geq F_2^{[a_t,b_t]}$ then agent 2 will take the portion $[a_t, x_t]$ and therefore agent 1 obtains a payoff $u_1(x_t, b_t) < u_1(a_t, F_1^{[a_t,b_t]}) < u_1(a_t, R_{[a_t,b_t]})$.

We prove now that the strategy T2 is a best response to strategy T1 in all subgames. We consider the following three cases:

(i) $X_1^t = [x_t, b_t]$ and $x_t \leq F_1^{[a_t,b_t]}$. T2 strategy prescribes that agent 2 takes the portion $[x_t, b_t]$. Note that the allocation $([a_t, x_t]\,(x_t, b_t])$ is efficient and that agent 1's utility level is lower than her half-cake equivalent utility. If agent 2 deviates, he can either take the portion $[a_t, x_t]$,which is obviously a less valuable portion than $(x_t, b_t]$, or make a different proposal. But then agent 1 according to T1 at step 3 chooses an action which is a best response in all subgames. By Lemma 3 agent 1's utility is equal or higher than the half-cake equivalent utility. Therefore by proposing any allocation agent 2 lowers his payoff.

(ii) $X_1^t = [x_t, b_t]$ and $x_t > F_1^{[a_t,b_t]}$. According to strategy T2 agent 2 proposes $Y_1^t = \left[a_t, F_1^{[a_t,b_t]}\right]$ if $u_2(F_1^{[a_t,b_t]}, b_t) \geq u_2(a_t, x_t)$, takes the portion $[a_t, x_t]$ otherwise. First note that in case agent 2 proposes $Y_1^t = \left[a_t, F_1^{[a_t,b_t]}\right]$ according to strategy T1 agent 1 takes the portion $\left(a_t, F_1^{[a_t,b_t]}\right)$ and therefore agent 2 receives the portion $(F_1^{[a_t,b_t]}, b_t]$. If agent 2 deviates from strategy T2, then either takes the portion $[x_t, b_t] \subset (F_1^{[a_t,b_t]}, b_t]$ or makes a different proposal. But then agent 1's utility is strictly higher than her half-cake utility level and therefore agent 2's utility will be lower than $u_2(F_1^{[a_t,b_t]}, b_t])$.

(iii) $X_1^t = [a_t, x_t]$. According to strategy T2 then agent 2 proposes $Y_1^t = [a_t, y_t]$ such that $u_1(Y_2^t) = u_1(a_t, R_{[y_t,x_t]})$ if $u_2(a_t, x_t) \leq u_2(x_t, b_t) + u_2\left(R_{[y_t,x_t]}, x_t\right)$, takes the portion $[a_t, x_t]$ otherwise. Note that $Y_1^t = [a_t, y_t]$ is the proposal, given strategy T1 and Lemma 7, that maximizes agent 2 payoff's when he decides to continue the game. Agent 2 decides to end the game if and only if the portion $[a_t, x_t]$ provides to him a higher payoff than the utility of the final payoff induced by his best proposal, and therefore strategy T2 is a best response.

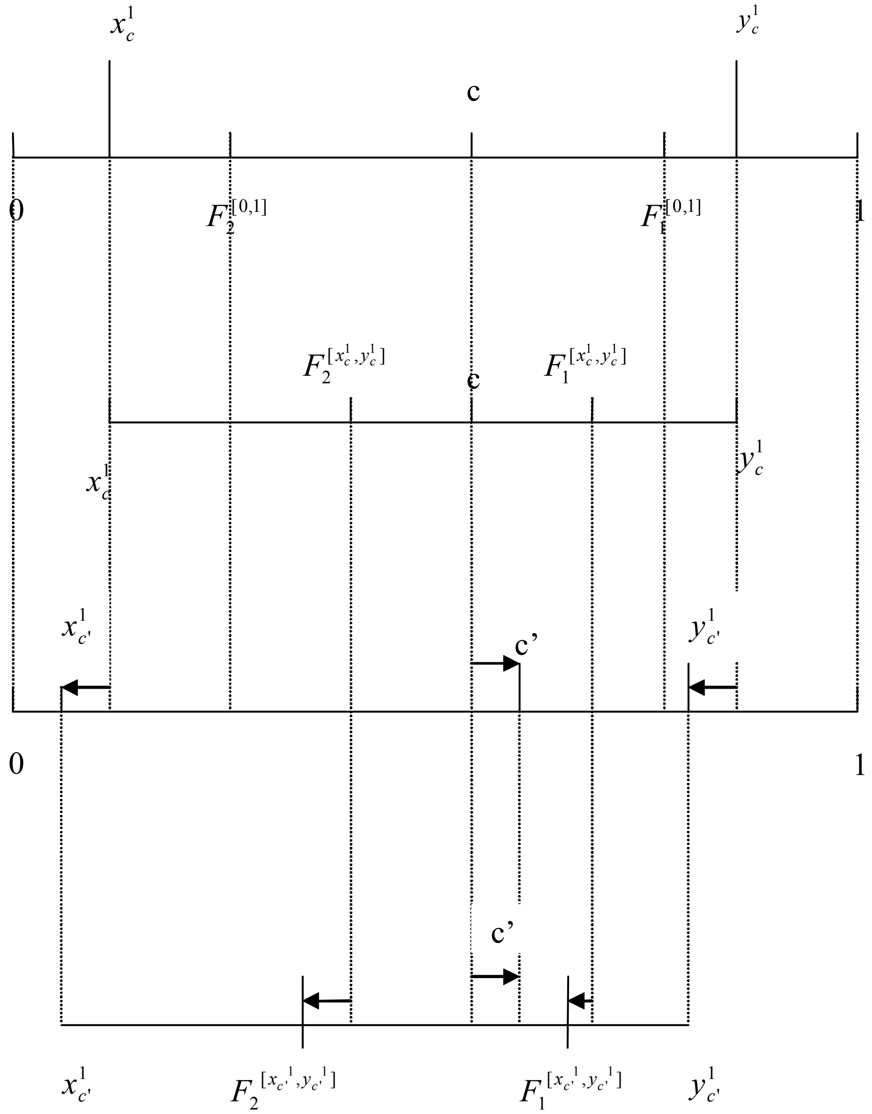

Figure 1: x_c^1 , y_c^1, $F_1^{[x_c^1,y_c^1]}$,and $F_2^{[x_c^1,y_c^1]}$ are decreasing in c

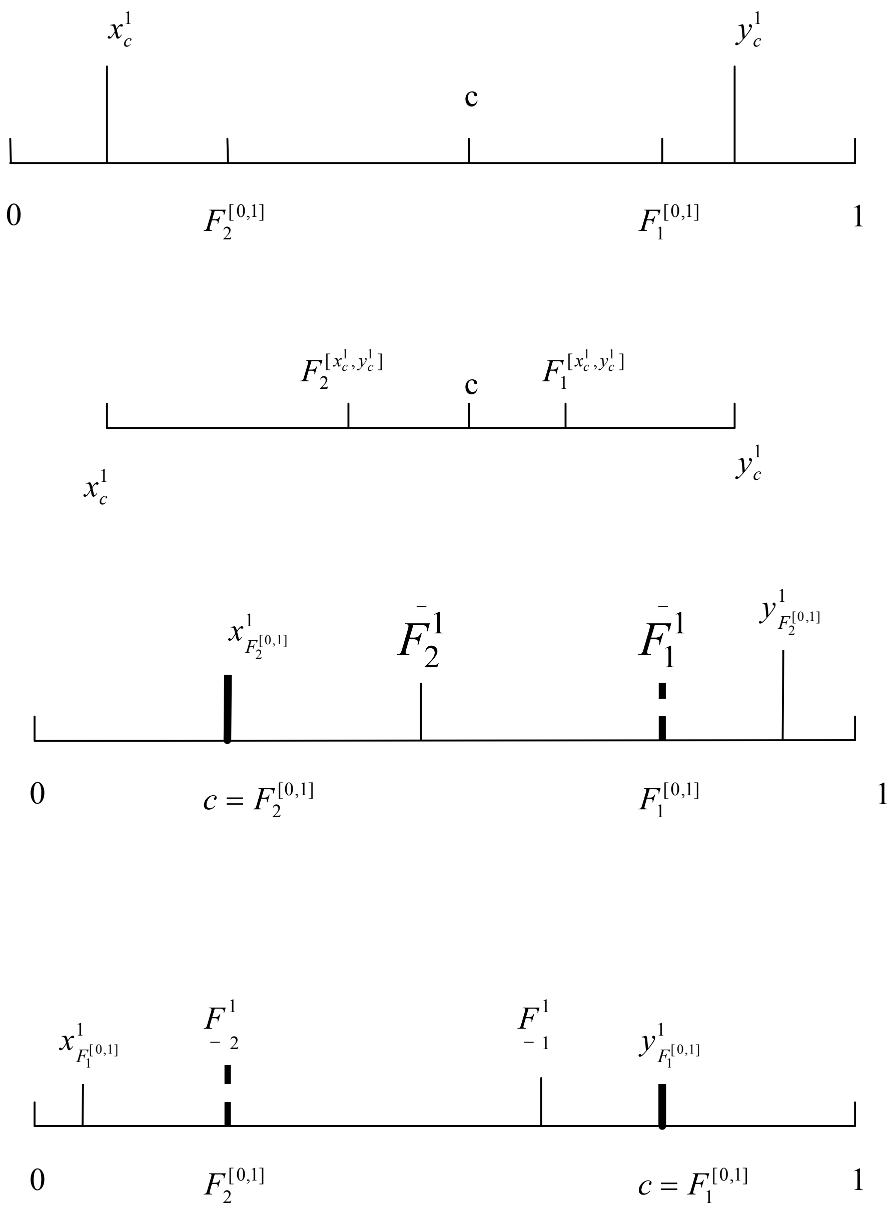

Figure 2: $\overline{F}_1^1 = F_1^{[0,1]}$ and $\underline{F}_2^1 = F_2^{[0,1]}$